Dynamic Demand Estimation in Auction Markets*

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Abstract

We study demand estimation in a large auction market. In our model, a dynamically evolving population of buyers with unit demand and heterogeneous and privately known preferences for a finite set of differentiated products compete in a sequence of auctions that occur in discrete time. We define an empirically tractable equilibrium concept in which bidders behave as though they are competing with the stationary distribution of opposing bids, characterize bidding strategies, and prove existence of equilibrium. Having developed this demand system, we prove that it is non-parametrically identified from panel data. We extend the model to consider a random coefficients demand system akin to workhorse demand models in industrial organization, and show that this too is non-parametrically identified. We apply the model to estimate demand and show how large sellers can exercise market power by using persistent reserve price policies, which induce higher bids and therefore revenues. Our analysis highlights the importance of both dynamic bidding strategies and panel data sample selection issues when analyzing these markets.

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1 Introduction

Most goods and services are sold at fixed prices. Yet auctions are used as the allocation mechanism in a wide variety of contexts, including procurement contracts, treasury bills and the granting of oil drilling and spectrum rights. Technology companies such as Facebook, Google and Microsoft sell advertisers access to online consumers through display and search advertising auctions. And though the majority of eBay's revenue now comes from fixed price listings of new goods, they still sell a large number of goods (both in absolute and dollar terms) by auction.

These markets share many common features. Buyers get multiple purchase opportunities over time, either for exactly the same product (e.g. a keyword in online search advertising), or for close substitutes (e.g. in treasury bill auctions). This allows bidders to intertemporally substitute, making participation and bidding decisions in light of the option value of waiting for future purchasing opportunities. Both bids and participation choices will reflect individual-specific preferences over the heterogeneous products available. For example, in highway procurement, Lewis and Bajari (2011) document matching between contractors and contract based primarily on distance and contract size, while in online labor markets, employers are more likely to award contracts to workers from their own country (Krasnokutskaya et al., 2016).

In the fixed price context, an influential discrete choice demand estimation approach has been developed that models how buyers with unit demand and heterogeneous preferences over item characteristics make purchasing decisions (Boyd and Mellman, 1980; Cardell and Dunbar, 1980; Berry, 1994; Berry et al., 1995).[1] However with the important exception of Jofre-Bonet and Pesendorfer (2003), there has been no analogous work in the empirical auctions literature, which typically considers the identification and econometric analysis of a repeated cross-section of observations of a static auction game with a single product. The goal of this paper is fill this gap.

Specifically, we consider a large-market model of repeated second-price auctions of differentiated products. Buyers have unit demand and heterogeneous, perfectly persistent and privately known multidimensional valuations (which may be formulated as latent random

[1]Goettler and Gordon (2011) and Gowrisankaran and Rysman (2012) extend these models to consider cases where consumers purchase repeatedly over time, and must form beliefs about future product offerings. By contrast we fix the set of products and impose unit demand in the present paper, as in the older literature.

coefficients over item characteristics). Their optimal bidding strategy is to shade their bids below their valuations to account for the option value of staying in the market and possibly purchasing a better good or the same good at a lower price in the future. We prove the existence of a dynamic equilibrium, and show non-parametric identification of the model. We then apply these results in estimating the demand for compact cameras on eBay.

There are three technical roadblocks we overcome. The first is that allowing for persistent privately known preferences in a dynamic auction game introduces private monitoring, with all its attendant game theoretic complications. We surmount this using a large market equilibrium concept, defining an equilibrium as a symmetric bidding function and set of beliefs about the distribution of rival bids such that the bidding function is optimal given those beliefs, and the beliefs are consistent with the ergodic distribution of equilibrium play.[2]

The second is a selection problem. Recall that we want to learn the distribution of the *vector* of valuations of bidders for the goods in the market. Even if bidders bid their valuations—and they don't—it would still be hard to learn the distribution of valuations, because most of them will exit, either randomly or by winning an auction, well before bidding on every available product and thereby revealing their valuations. Moreover, auction winners are disproportionately those with high valuations, which introduces a selection problem. We show that we can re-weight the observed distribution of these different types of bidders to learn their "bid-type": their preferred bid vectors.

The final piece of the puzzle is learning an inversion from bid-types to valuations. A useful insight is that a bidder with a given valuation vector faces a Markov decision problem (MDP) when deciding how much to bid on each product, where the payoff functions and transition matrices are observed. We show that the MDP can be inverted: if we know a bidder's bid-type—their optimal bid on each product—we can invert it to get their underlying type, or valuation.[3] By combining this inversion with the selection correction, we can identify the distribution of valuations.

A subsequent theorem proves non-parametric identification for a random coefficients demand

[2]This is similar to ideas elsewhere in the literature, such as the model of belief formation in Krusell and Smith (1998) and the oblivious (mean-field) equilibrium concept of Weintraub et al. (2008) and Iyer et al. (2014). Many other papers in the dynamic games estimation literature have instead assumed that everything persistent is commonly known, but this is counter to the spirit of our project (for examples, see e.g. Jofre-Bonet and Pesendorfer (2003), Aguirregabiria and Mira (2007), Bajari et al. (2007), Pakes et al. (2007); Fershtman and Pakes (2012) is a recent exception.)

[3]A similar argument was made in Jofre-Bonet and Pesendorfer (2003) and subsequently in many of the papers cited in the literature review below.

system, paralleling the workhorse demand model in industrial organization, where the utility function combines latent and persistent random preferences over product characteristics, transient unobserved heterogeneity draws that are common to bidders in an auction, and transient idiosyncratic preference shocks for individual bidders. This theorem makes use of a novel deconvolution argument based on repeated observations of the same bidder's bids in successive auctions.

Our setup endeavors to situate auctions in a *marketplace*, with opportunities for substitution that are often missing in the auctions literature. However, in reality, substitution is much richer than any one model will capture: it happens through search; it happens as bidders choose which of concurrent auctions to enter, and it also happens intertemporally, as bidders eye future opportunities. We have deliberately chosen to focus on the latter channel. In our setting, a bidder with a preference for a particular type of product will shade their bid aggressively until it is auctioned, effectively substituting inter-temporally rather than through search or entry. This loads substitution into the continuation value, which we show to be quite tractable for empirical modeling.

The final part of the paper is an application of our framework to data from eBay's compact camera market. We model preferences as a linear combination of a random taste for a camera and a random taste for camera resolution, and estimate the distribution of these random coefficients. Our estimation approach uses only the first-and-second highest bids in each auction, as it is hard to know how to interpret other bids. We are able to accommodate this data limitation by modifying our selection correction argument, so that the censoring of bids that were neither first nor second highest is explicitly accounted for, which illustrates the flexibility of our approach. Despite this, we get statistically precise estimates of demand, and match out-of-sample moments of the bid distribution dataset quite well.

We use these results to consider how a large seller, controlling approximately one third of listings, might exercise market power in an auction marketplace. By committing to a persistent reserve price policy (i.e. one that applies to all listings from this seller, versus a transient reserve, which is implemented in only one auction), the seller can reduce buyers' continuation values, which translates into more aggressive bidding and higher revenue. Simulation using our demand estimates shows that this effect can be quite large. When the seller's costs are 90% of the expected sale price under no reserves they are able to use optimal persistent reserves to raise profits by as much as 42%, versus 34% under a transient reserve. This difference stems from the market power of the seller, and will grow as their market share—and therefore their ability to depress bidders' continuation values—does.

Literature Review. Jofre-Bonet and Pesendorfer (2003) was the first paper to attack estimation in a dynamic auction game in the context of sequential procurement auctions. Subsequent to this, a number of papers have looked at dynamics on the eBay platform specifically. Zeithammer (2006) developed a model with forward-looking bidders, and showed both theoretically and empirically that bidders shade down current bids in response to the presence of upcoming auctions of similar objects. Sailer (2006) estimates participation costs for bidders facing an infinite sequence of identical auctions (see also Groeger (2014) for a dynamic model with participation costs in a procurement environment). Nekipelov (2007) analyzes a model where bidders attempt to prevent learning by late bidding. Ingster (2009) develops a dynamic model of auctions of identical objects, and provides equilibrium characterization and identification results. Finally, Ridinger (2020) incorporates resale and aggregate shocks into a dynamic model that is applied to the market for classic cars.

The original version of our paper, Backus and Lewis (2010), introduced a dynamic auction model with latent, persistent and high-dimensional types, and showed that under certain restrictions on player beliefs, equilibrium existence and non-parametric identification could be proved. Subsequently, there have been two other notable papers about dynamic auction markets, and eBay specifically. Bodoh-Creed et al. (2017) estimate demand for Kindles, and then simulate an alternative market design where half as many 2-unit auctions were run instead, finding that this would increase efficiency but lower seller revenues. Their model uses the same "large market" belief approximations as our original paper, but they go further by employing a continuum approximation with an infinite number of bidders and products that they show behaves similarly to the finite game and has certain computational advantages. They also allow for participation costs and endogenous entry. Hendricks and Sorensen (2014) keep the finite bidder model and "large market beliefs", but do the analysis in continuous time, and again allow for selective entry: bidders pick which of an upcoming set of available auctions to participate in, depending on the current state (e.g. current posted bids in these auctions). They use their estimates to understand how well the marketplace works relative to a fully efficient competitive benchmark. The main difference between these papers and ours is that their models are of homogeneous goods and focus on efficiency and market design, whereas we have heterogeneous objects and are interested in substitution between products.[4]

[4]Other papers tackle related issues in a static setting. Gentry and Li (2014) and Adams (2012) offered

2 The Demand System in Context

We begin the paper with a brief review of the relevant portion of the empirical auctions literature and how it relates to similar work on fixed price markets. The starting point of this literature is a model where the payoff of a winning bidder takes the form:

$$u_i = \varepsilon_i.$$

Specifically, short-lived agents i draw independently and identically distributed (iid) real-valued valuations ε_i from some distribution F, and participate in some auction mechanism to win the goods (see e.g. the work of Guerre et al. (2000) on first-price auctions). The econometrician observes a repeated cross-section of bids from this mechanism and wants to learn F. Subsequent elaborations of this work included treatments of non-iid and asymmetric valuation distributions, incomplete data and other auction mechanisms (see Athey and Haile (2007) for a discussion of identification in these models).

Maintaining the static auction environment, but adding multiple distinct items with characteristics that are valued equally by all participants yields a utility function of the form:

$$u_{ij} = \alpha' z_j + \varepsilon_{ij}$$

where items are indexed by j, z_j is a vector of item characteristics and α is an unknown parameter vector. Haile et al. (2006) demonstrate how to estimate the α in a first-stage "normalization" step prior to learning the distribution F. Notice the resemblance to the utility function assumed in a standard multinomial logit.

Moving to a dynamic context in which a fixed set of long-lived bidders repeatedly interact over an infinite horizon, Jofre-Bonet and Pesendorfer (2003) (JBP) consider a more general model in which the per-period payoff of winning bidders u_{ijt} is sampled iid from a conditional distribution $F|s_t$, where the state variable s_t can include item characteristics z_j, individual characteristics α_i and even the characteristics of rival bidders α_{-i} (in their application to highway procurement, z_j are the contract size and duration, and α_i and α_{-i} are the size and

analyses of non-parametric identification in models of costly endogenous buyer entry and optimal entry with multiple competing auctions, respectively, while Marra (2018) models both buyer and seller entry in order to study fee–setting on two–sided auction platforms. And while we focus on heterogeneity in preferences on the buyer side, Krasnokutskaya et al. (2014) consider a world with heterogeneous preferences on the seller side.

backlog of existing projects by that bidder and rival bidders respectively). The full state vector s_t is assumed to be known to the econometrician. An illustrative special case of the model is an index model with preferences over item characteristics that vary with bidder observables:

$$u_{ijt} = \alpha'(w_i \times z_j) + \varepsilon_{it}$$

where w_i is a column-vector of observable bidder characteristics, z_j is a column-vector of observable item characteristics, $w_i \times z_j$ is notation for the column-vector consisting of all pairwise interactions of w_i and z_j (equivalently a flattened version of the outer product), α is an unknown parameter vector and ε_{it} has unknown distribution F. Again, notice the resemblance to a multinomial logit demand system.

In the current paper, we study a dynamic equilibrium in the still more general case of random coefficients, developed by Boyd and Mellman (1980) and Cardell and Dunbar (1980), and well-known in the IO literature due to the work of Berry et al. (1995):

$$u_{ijt} = \alpha_i z_j + \xi_t + \varepsilon_{it}$$

where now α_i is a person-specific, perfectly persistent and latent random variable with dimension equal to the number of product characteristics ("bidder type"), ξ_t is an auction-specific transitory valuation shock and ε_{it} is an idiosyncratic valuation shock. The goal becomes to learn the distributions of α_i, ξ_t and ε_{it}. We also consider a less heavily parameterized version of the model in which consumers draw iid valuations $\mathbf{x}_i \in \mathbb{R}^J$—a real-valued valuation for each kind of product—and the goal is to learn the distribution of such valuations F.

The main difference between our model and JBP is that we allow for a multidimensional, persistent and *latent* state vector consisting of the bidder types. So for example we can allow for persistent productivity differences between highway construction firms that cannot be perfectly explained by observables such as the current portfolio of projects and project backlog. In our application to eBay, because we allow for persistent latent types, we can model consumer preferences as stable. A smaller difference is that we also allow for correlated taste shocks within an auction, through the unobserved heterogeneity term ξ_t. This is specifically ruled out in JBP. A third difference is that we assume the state is not commonly known by either the bidders or the econometrician (i.e. the bidders do not know each other's valuations). So whereas in JBP bidders know the distribution of competing types, in our model that requires sophisticated inference, creating game theoretic complications that we circumvent by the use of large market arguments. The next section presents the formal model and these arguments.

3 Model and Equilibrium Analysis

We consider a market in which competing products are sold by second-price sealed bid auctions. These auctions are held in discrete time, with either zero or one good auctioned per period over an infinite horizon. Since our focus is on demand, we assume for simplicity that supply is random and exogenous. Bidders have unit demand, and enter the market with private and perfectly persistent valuations for each of the objects. They are inattentive, and are active and bid in any particular period with constant probability. We show that they bid their valuation less their continuation value, and assume they assess the latter based on the steady-state distribution of supply and competing bids, rather than on current market conditions (e.g. the number of competing bidders in the current auction). Winning bidders exit the market with certainty, while losing bidders exit with constant probability.

We have chosen this set of assumptions to match some features of the market for digital cameras on eBay, which is our empirical application. In any eBay category (such as digital cameras), there are many different products sold by auction to a large number of anonymous buyers. Although these auctions typically last for many days, and thus overlap—so that at any given point in time there are many auctions occurring simultaneously—they finish at different ending times, in sequence. As Bajari and Hortaçsu (2004) and Hendricks and Porter (2007) have noted, this timing, combined with the way the proxy bidding system works, imply that eBay is reasonably well approximated by sequence of second-price sealed bid auctions.

3.1 Environment

We formalize the above description of the environment in what follows:

Supply. There are J distinct kinds of goods sold in a market, indexed by $j = 1 \dots J$. We denote the set of products by $\mathcal{J}$. In each period t, a good may be available for purchase. Supply is exogenous and Markov, with the current product j_t drawn from a stationary multinomial distribution conditional on the lagged product j_{t-1}. We allow for the possibility that no product is available in a given period, and so supply can be summarized by a square transition matrix Q of size $J+1 \times J+1$, where the entry $Q_{j,k}$ gives the probability product k will be supplied next when j is currently offered (and the last row and column give the cases where no product is offered now and will be offered later, respectively). We assume moreover that the multinomials have full support, so that regardless of what was supplied at $t-1$, it is possible that any of the J products (or nothing) is supplied at t. When a good is available, it is sold by second-price sealed-bid auction.

Demand. At the beginning of each period, E_t buyers enter the market, where E_t is sampled independently over time from a Poisson distribution with mean λ. Each buyer has unit demand, and has a *perfectly persistent* value for each of the goods summarized by a privately known vector of valuations $\mathbf{x} = (x_1, x_2 \cdots x_J) \in \mathbb{R}^J$. This value is drawn iid across buyers on entry according to a distribution $\mathbf{F}$ with strictly positive density over its support $\mathcal{X} = [0, \overline{x}]^J$. The new entrants combine with the population of buyers from previous periods to form a cohort of bidders of size N_t.[5]

New entrants are always active; the remaining incumbent members of the cohort are active in each period with iid chance τ. Active buyers participate in the current auction (if one is held), observing the product currently under auction, and placing bid $b \in \mathbb{R}$.[6] Notice that we allow for negative bids. This plays an important simplifying role in the identification arguments below—since otherwise we would have to deal with censored data—but is not important for the theory presented in this section.[7] Agents are risk-neutral and have quasi-linear utility, receiving a total payoff of $x_j - p$ for buying a good j at price p, and zero otherwise. If an auction is held and a bidder wins the auction, they exit the market. All other active bidders exit with probability $(1-r)$ (i.e. $r \in (0,1)$ is the survival rate). Inactive bidders do not exit. We assume that agents do not discount future payoffs, although the exogenous exit probability is functionally equivalent to exponential discounting.[8]

[5]For the proof of Lemma 6, we also need to assume that there exists some arbitrarily large maximum number of buyers $\overline{N}$ such that when $N_t = \overline{N}$, no-one enters ($E_t = 0$). This is a purely technical condition required for compactness of the state space.

[6]Note that this rules out an explicit model of search, where buyers endogenously choose which of an upcoming sequence of auctions to bid in. This is a limitation of the model. See Hendricks and Sorensen (2014) for a way to account for this.

[7]If bids must be positive, bidders will simply not participate whenever their desired bid is negative. Their continuation values must be adjusted accordingly.

[8]For this reason the agents' time preferences are not separately identified in this setting.

3.2 Beliefs and Strategies

We begin our analysis by looking at the buyer's beliefs. In our application to the compact camera market on eBay, it seems reasonable to assume that buyers have simple models of the competition they face. We formalize this idea by assuming that bidders believe that the distribution of the highest competing bid is equal to the historical average and best respond accordingly.[9] Let B_j^1 be a random variable denoting the highest bid in an auction for good j, and let G_j^1 be its cumulative distribution function.

Assumption 1 (Beliefs About Competing Bids). *Following any history of play, bidders believe that the highest rival bid in an auction for good j has distribution G_j^1.*

These are rational beliefs to hold: an uninformed bidder, entering a stationary market, should expect the distribution of competing bids to look like the historical distribution of bids. This is because their own entry into the market is random and thus uninformative as to the current state, which means the distribution of rival bids and historical bids coincide.[10] What the assumption rules out is signaling and learning, e.g. that bidders are more sophisticated, updating their beliefs based either on their own experience after entering the market or from publicly available state variables such as the sequence of upcoming auctions.[11] Bidders are "oblivious" in the sense of Weintraub et al. (2008) (though their oblivious equilibrium concept applies only to games of complete information).

We make this modeling choice to simplify the game theory, but it is also attractive as a model of bidder behavior. Being fully rational and conditioning on all available information is presumably costly, and the gains are small. This is because under repeated second-price auctions, the optimal bid today depends on the current state of the market only to the extent that it predicts future competition (the payoff on losing), and the market turns over quickly enough that the current state is not particularly informative. At the end of this section, we

[9] Some version of this assumption is made throughout the related literature (Ingster, 2009; Bodoh-Creed et al., 2013; Hendricks and Sorensen, 2014).

[10] Contrast this with a repeated static auction model with N bidders drawing iid valuations; then the historical highest bid distribution G^1 is the maximum of N bids, whereas the highest rival bid distribution is the maximum of $N-1$ bids. When the number of bidders is fixed, the fact that a bidder is in the auction is informative: it means that only $N-1$ people are competing with them. Here there is no fixed number of bidders and the bidder's presence does not inform their beliefs about G^1.

[11] A bidder who lost yesterday might reason that competition was fierce yesterday, and may still be so today. This gives rise to a winner's curse effect (Budish, 2008). There is some evidence that this kind of learning occurs on eBay: Coey et al. (2015) show that bidders tend to (modestly) increase their bids in subsequent auctions for the same product, although they attribute this to deadlines rather than learning.

outline an extension that allows bidders to condition on other public state variables.

Under this assumption, the only payoff relevant-information at time t is the bidder's type and the current object under auction, j_t. So without loss of generality a pure strategy is just a vector $\beta : \mathcal{X} \to \mathbb{R}^J$, a bid on each product as a function of type, with jth coordinate β_j. In the paper, given the number of agents typically seen on eBay, we choose to restrict attention to symmetric strategies.

3.3 Equilibrium

In an equilibrium of this market, bidders will play optimally given their beliefs about the competition they face. Those beliefs are informed by the long-run distribution of competing bids. So in order to make progress in defining an equilibrium, we need to talk about the long-run evolution of the market.

The environment evolves as a first-order Markov process. To see this, let the state ω_t be a real-valued vector of dimension $\overline{N}+1$, consisting of first the product-type to be auctioned that period (-1 if no product is available that period), then the valuations of the incumbent bidders in the next N_t positions (including bidders who have just entered) and -1 entries in the rest of the positions. Then next period's state depends on who wins and who loses (a function of the current state and strategies β, as well as who is active that period), which losers survive, what product is sold next period and the number and valuations of new entrants. Let $T(\mu, \beta)$ be a transition operator that takes a distribution μ over current states ω into distributions over the next period's distribution of states ω'. Then for each β, a corresponding erdogic measure is a fixed point: $\mu = T(\mu, \beta)$.

Using this language, we can define an equilibrium:

Definition 1 (**Equilibrium**). *A pure strategy equilibrium is a tuple* $(\beta^e, \{G_j^1\}, \mu^{\beta^e})$ *such that*

(Optimality) $\beta^e(\mathbf{x})$ *is a best response for type* $\mathbf{x}$ *given beliefs* $\{G_j^1(b)\}_{j\in\mathcal{J}}$*;*

(Ergodicity) μ^{β^e} *is the unique fixed point of the transition operator* $T(\mu, \beta^e)$*, and*

(Consistency) $G_j^1(b) = \int 1\left(B_j^1(\omega) \le b\right) d\mu^{\beta^e}(\omega|j)$*, where* $\mu^{\beta^e}(\cdot|j)$ *is the conditional equilibrium ergodic measure given that product* j *is being auctioned.*

The definition mirrors the requirements for a Bayes–Nash equilibrium: play must be optimal given beliefs, and beliefs must be consistent with play. The twist is that here we link strategies to beliefs through the ergodic measure μ^{β^e}. The ergodicity requirement defines μ^{β^e} to be the unique fixed point of the transition operator when the strategies are β^e, i.e., what will happen in the long-run if β^e is played. The consistency requirement says that beliefs about competing bids, as specified by Assumption 1, must be consistent with the equilibrium ergodic measure. Notice that the uniqueness requirement on the ergodic measure is necessary otherwise beliefs are not well-defined.

3.4 Best Responses

We find best responses through solving the buyer's decision problem. This problem is dynamic, since losers may have an opportunity to bid at a later date. Each buyer of type $\mathbf{x}$ must solve a dynamic program where the state is the product–type currently being auctioned, since Assumption 1 implies that this is sufficient for beliefs about the highest competing bid (which together with the bid determines allocations and payments). Since the state is Markov, the buyer faces a Markov decision problem (MDP).

Define the state transition matrix $\widetilde{Q} \equiv \sum_{s=1}^{\infty} \tau(1-\tau)^{s-1} Q^s$. This is the distribution over products a bidder bidding on j today expects to see, determined by Q^s when they are next active, as determined by τ. Let $v_j(\mathbf{x})$ be the perceived continuation value of a bidder of type $\mathbf{x}$ who is active and bidding on product j. Then we can write down a Bellman equation:

$$v_j(\mathbf{x}) = \max_{b \in \mathbb{R}^+} \int \left(1(b \geq B_j^1)(x_j - B_j^1) + 1(b < B_j^1) r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}) \right) dG_j^1(B_j^1). \tag{1}$$

The first term in the integral represents the case where the bidder submits the largest nonzero bid, winning the auction and obtaining surplus equal to current valuation less a payment given by the second-highest bid (potentially zero). The second term represents the case where the bidder loses and survives to bid another day, obtaining their continuation value for the next period in which they will be active. These events are determined by the realization of the highest competing bid, which by Assumption 1, the bidder believes to be distributed according to G_j^1. Solving the above maximization problem, we get:

Lemma 1 (**Best Responses**). *Suppose that beliefs satisfy Assumption 1. Then bidding* *valuation less discounted continuation value is a best response:*

$$\beta_j(\mathbf{x}) = x_j - r \sum_k \widetilde{Q}_{j,k} v_k(\mathbf{x}), \tag{2}$$

where the dependence on the distributions $\{G_j^1\}$ is through the continuation value. When G_j^1 is supported on the interval $[0, \bar{b}_j]$ (i.e., has no gaps), $\beta_j(\mathbf{x})$ is the unique best response. Moreover, $\beta_j(\mathbf{x})$ is continuous, strictly increasing in x_j and decreasing in x_k for $k \neq j$.

Because this is a second-price auction, bidders bid to be indifferent between marginally winning and losing. Winners who marginally win receive the difference between their value and their bid, while losers receive their continuation value (discounted to account for the possibility of exit). Equating these, bids must be equal to value less continuation value. This optimal bid might sometimes be negative, indicating that a bidder would optimally wait for another product rather than win the current product at a positive price.[12]

The characterization in (2) is implicit—i.e., $v_j(\mathbf{x})$ is defined recursively according to the Bellman equation in (1). But since the state space of the MDP is discrete, and the total value of the game for $\mathbf{x}$ is upper bounded by the maximum valuation $\max_j x_j$ (a bidder's payoff if they win and pay zero, exiting the market), the MDP has a solution and a unique value function (Blackwell, 1965). Since optimal bids are given by valuations less continuation values, the uniqueness of the value function implies the uniqueness of $\beta_j(\mathbf{x})$ whenever G_j^1 is supported on an interval.[13] Monotonicity is a natural property: if a bidder values object j more, they optimally increase their bid for it, and correspondingly shade their bids on substitute objects down.

3.5 Existence

To tie up this model, we state the first main result of the paper, an existence theorem:

[12]Similar equations appear in many other papers. As Budish (2008), Said (2009) and Zeithammer (2009) all note, this bid characterization is not correct in Bayes-Nash equilibrium, due to a winner's curse effect: on winning, a bidder learns that the remaining types had lower valuations, and therefore regrets winning now rather than later at a lower price. This effect disappears when private valuation shocks are transient (e.g. Jofre-Bonet and Pesendorfer (2003)) or under large market assumptions on beliefs (our Assumption 2 or those made in Iyer et al. (2014), Hendricks and Sorensen (2014) and Bodoh-Creed et al. (2017)).

[13]The requirement on G_j^1 is used to rule out the possibility of an interval $[b, b']$ on which rival bids are unsupported, since in that case if the putatively unique best response falls in this range, any other bid in the range will also be a best response.

Theorem 1 (Existence). *A pure strategy equilibrium exists in which bidders bid according to $\beta_j(\mathbf{x})$ as defined in Lemma 1. When $J = 1$, the equilibrium is almost everywhere unique.*

To prove the result, we first establish in a lemma that for any symmetric strategy β, a corresponding unique ergodic measure exists. The existence of an ergodic measure follows from compactness of the state space: there are a finite number of players, all with bounded valuations, and a finite set of products to be sold. Uniqueness follows by showing that all states communicate with one another, ruling out the possibility that the initial state determines the ergodic measure.

Next we consider the single product environment. When there is only one good, the types are totally ordered by their value for that good. Since equilibrium bids must be strictly increasing in valuations (see Lemma 1), all equilibrium strategies generate the same set of winners and losers each period (the highest active type wins, the other active bidders lose), and hence the same ergodic distribution of types. This implies (via the consistency requirement) that equilibrium beliefs about opposing types are invariant to the strategy. This simplifies the analysis considerably, allowing us to prove uniqueness via a contraction-mapping argument.

In higher dimensions though, different strategies β may lead to different ergodic measures. We must show that there exist equilibrium strategies β^e that are a best response given beliefs $\{G_j^1\}$ that are consistent with the ergodic measure μ^{β^e}. We prove this by applying Schauder's fixed point theorem on the space of continuous functions on the type space. The proof proceeds by showing that the set of best responses to any continuous bidding function is uniformly equicontinuous and bounded (a compactness condition), and that the best response varies smoothly with the bidding function (a continuity condition). To establish continuity, we prove in Lemma 7 in the appendix that the ergodic measure is continuous in the strategies, so that beliefs and best responses are continuous too.

3.6 Extension: State-contingent beliefs and valuations

Up to now, we have assumed that bidders have private perfectly persistent valuations, and treat the distribution of opposing bids as fixed. We would like to allow for a more flexible demand system that incorporates random coefficients, unobserved heterogeneity and idiosyncratic shocks, as in Berry et al. (1995). We would also like to allow for the possibility that bidders condition on some publicly observable state variables—e.g. the number of upcoming auctions on a given product in the next hour—when forming beliefs about rival bids.

We formalize this by enriching the state space. Assume that in each period, a state variable $s \in \mathcal{S}$ is publicly observed. This state variable may affect bidder valuations, so we model a bidder's valuation as a deterministic function $x(s)$ of the random state s. Exactly how s is sampled will depend on the specifics of the model. For example, in the original model, the state variable is the product-type under auction (i.e. $s \in \mathcal{J}$). When $s = j$, their value is x_j.

The state variable may also affect beliefs about competing bids, so we modify Assumption 1 accordingly:

Assumption 2 (**Beliefs with Public States**). *When the public state is s, bidders believe (i) that the highest rival bid in the current auction has distribution $G^1(\cdot|s)$ and (ii) will transition according to the exogenous Markov transition kernel $P(s'|s)$.*

For example, bidders may believe that when there are many upcoming auctions for a given good, the highest competing bid will be lower.

Definition 2 (**Equilibrium with Public States**). *A pure strategy equilibrium is a tuple $(\beta^e(x(s), s), \{G^1(\cdot|s)\}_{s\in S}, \mu^{\beta^e})$ such that*

(Optimality) *$\beta^e(\mathbf{x})$ is a best response in state s for type $x(s)$ given beliefs $G^1(b|s)$ $\forall s \in \mathcal{S}$;*

(Ergodicity) *μ^{β^e} is the unique fixed point of the transition operator $T(\mu, \beta^e)$, and*

(Consistency) *$G^1(b|s) = \int 1\left(B^1(\omega) \leq b\right) d\mu^{\beta^e}(\omega|s)$ where $\mu^{\beta^e}(\cdot|s)$ is the conditional equilibrium ergodic measure given the state is s.*

Then by the same logic that we offered for Lemma 1, it is optimal for bidders to bid their state-contingent valuation less their discounted continuation value conditional on the current state, and we obtain:

Lemma 2 (**Best responses with Public States**). *Suppose that beliefs satisfy Assumption 2. Then the best response function $\beta(x(s), s)$ satisfies:*

$$\beta(x(s), s) = x(s) - r\int v(s')dP(s'|s) \tag{3}$$

where $v(s)$ is the value function, defined according to the Bellman equation:

$$v(x(s), s) = \max_{b\in\mathbb{R}^+} G^1(b|s)(x(s) - E[B^1|B^1 < b, s]) + r(1 - G^1(b|s))\int v(x(s'), s')dP(s'|s). \tag{4}$$

4 Identification

We now turn to non-parametric identification of this dynamic auction model, allowing for persistent private types, listing-specific and idiosyncratic shocks. We assume throughout this section that a single equilibrium is played by the bidders. The data takes the form of a panel, consisting of all bids and the products they bid on from every bidder who was active in the market. Each individual bidder time series will have gaps, corresponding to auctions in which they did not participate because they were not active in that period. The time series will also be of different lengths, since bidders will exit at different times. There are random and non-random reasons for exit: winners are likely to be types with high valuations, but among losers, exit is at random.

In the absence of additional shocks, each bidder has a persistent optimal bid for each good that we call their bid-type. We show that there is an inversion from bid-types to valuations, building on an argument first presented in Jofre-Bonet and Pesendorfer (2003). But only bidder time series that contain a bid on every product—"complete" bidder time series—can be inverted in this way, and due to endogenous exit the sample of complete bid vectors is selected. An important part of our identification argument is correcting for this selection. We present these inversion and selection correction steps initially through a two good example.

The following subsection generalizes to the case with item-specific observables and item-specific and idiosyncratic latent shocks. Since the persistent bid-types are no longer directly observed in the data (because of the shocks), we cannot invert pointwise from bids to valuations. Instead we work with distributions, showing that the distribution of complete bid vectors is a convolution of the distribution of bid types with the various shock distributions. A combination of deconvolution, selection correction, and inversion arguments suffices for identification.

4.1 Identification with two goods

There are two goods. The econometrician has access to a panel dataset with all bids placed and the type of good auctioned in each period. The generic steps of the identification argument are:

STEP 1: IDENTIFYING Q, λ, r, τ

Identification of Q is immediate, as the econometrician sees the sequence of products up for auction. Next, since the entry distribution is Poisson, λ is just the mean number of entrants each period. The survival rate r is identified by the probability that the a bidder ever returns to the market following a loss:

$$r = \mathbb{P}(\text{bidder i is observed in any period} > t|\text{bidder i bid and lost in period } t). \qquad (5)$$

Moreover, the probability that a bidder bids in the very next auction after losing is:

$$r\tau = \mathbb{P}(\text{bidder i is observed in period t + 1}|\text{bidder i bid and lost in period t}). \qquad (6)$$

From this pair of equations one can solve for the activity rate τ.

STEP 2: INVERSION

Next, we want to infer the type vectors $\mathbf{x} = (x_1, x_2)$ from the bid-vectors $\mathbf{b} = (b_1, b_2)$. We will call these bid vectors $\mathbf{b}$ *bid-types* and denote their distribution by $\widetilde{\mathbf{F}}$ with density $\widetilde{f}$. Mathematically, this is the distribution obtained by applying the bid function β to each type $\mathbf{x}$ pointwise. Define the matrix $\widetilde{Q} = \sum_{s=1}^{\infty} \tau(1-\tau)^{s-1}Q^s$. For a currently active bidder facing good j, row j of this matrix gives the probability distribution over the good that will be supplied the next time they are active.[14] In this example, $\widetilde{Q}$ is a 2×2 matrix with all entries equal to 1/2 since supply is independent across periods.

Next, define the function $\zeta : \mathbb{R}^J \to \mathbb{R}^J$ as

$$\zeta(\mathbf{b}) \equiv \mathbf{b} + r\widetilde{Q}(I - r\widetilde{Q})^{-1}G^1(\mathbf{b})\left(\mathbf{b} - E[\mathbf{B^1}|\mathbf{B^1} < \mathbf{b}]\right). \qquad (7)$$

This function ζ is the inverse bidding function. Optimal bids are valuations less continuation values (Lemma 1), and so valuations are bid-types plus continuation values. The second term on the RHS of (7) is the continuation value. As we prove later, it is equal to the stream of payments from a set of J annuities, where the jth pays $G_j^1(b_j)(b_j - \mathbb{E}[B_j^1|B_j^1 < b_j])$ whenever j is auctioned. The value of this stream of payments is the sum of a geometric series, and the pre-multiplication by $r\widetilde{Q}(I - r\widetilde{Q})^{-1}$ yields the continuation value.

The function $\zeta(\mathbf{b})$ can be constructed from the observable data, since the distributions of highest bids for each products $\{G_j^1\}_{j\in\mathcal{J}}$ are observed. So if we can measure the distribution

[14] The formula follows by taking a weighted average of the probability distribution over goods supplied at various horizons s, and multiplying by the probability of being first active at that horizon $\tau(1-\tau)^{s-1}$.

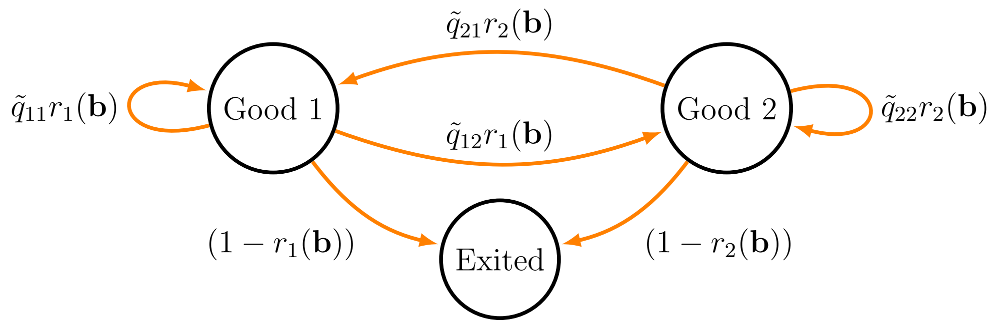


Figure 1: Bidder Path as a Markov Chain. The bidder starts by bidding on either good 1 or good 2, uniformly at random. From there they can either exit, bid on the same product again, or bid on the other product.

of bid-types $\widetilde{\mathbf{F}}$, we can recover the primitive distribution of types $\mathbf{F}$ from:

$$F(\mathbf{x}) = \mathbb{P}_{\widetilde{\mathbf{F}}}(\{\mathbf{b} : \zeta(\mathbf{b}) \leq \mathbf{x}\}). \tag{8}$$

Step 3: Correcting for Selection

We want to measure the distribution of bid-types $\widetilde{\mathbf{F}}$. But our data consists of individual bidder time series, and only some of them will be *complete*, by which we mean that they include a bid on both products. A complete bid series may have many entries, but only two will be distinct: a bid on product 1 and a bid on product 2. Collapse each complete bid series to this 2-vector, and let the joint density of these vectors in our dataset be $g(\mathbf{b})$, and the joint density of bid-types be $\widetilde{f}(\mathbf{b})$. We have a *selection problem*: $g(\mathbf{b}) \neq \widetilde{f}(\mathbf{b})$.

To see this, let $s(\mathbf{b})$ be a function that gives the probability that a particular bid-type will be observed bidding on both products. To calculate this selection probability, notice that we can view a bidder's path on the platform as a Markov chain. Figure 1 illustrates this Markov chain. A bidder starts off either by bidding on good 1 or good 2. With probability $r_j(\mathbf{b}) = (1 - G_j^1(b_j))r$ they do not exit following the auction (i.e. does not win and then survives). If they don't exit, the probabilities they bid on each of the goods when next active are given by the entries $\tilde{q}_{ij}$ of the $\widetilde{Q}$ matrix defined earlier. This illustration assumes that the probability of winning for any particular bidder is $G_j^1(b_j)$, *independent of the bidder's past bids and losses*, i.e., that bidders are small relative to the market. This assumption isn't necessary for identification, as we argue below, but it does help to make the state space manageable in applications.

Now, all of the transition probabilities are identified, since the distributions of highest bids

$\{G_j^1\}_{j\in\mathcal{J}}$, survival rate r, activity rate τ and the supply matrix Q are all identified. The selection probability $s(\mathbf{b})$ is the probability that a bidder who enters the market in one of the transitory states corresponding to a particular good visits every other transitory state before hitting the absorbing exited state. This probability can be computed in closed form, and thus $s(\mathbf{b})$ is a known function of identified quantities, and so is itself identified.

This selection argument can be modified to account for other forms of dynamic selection. For example, on eBay, we believe that only the top two bids in each auction may reasonably be used in estimation, generating a new selection problem (the bidders making the top two bids are a non-random set of types). This too can be modeled and thus addressed—see Appendix B.6 for details on our approach to selection in our empirical application.

So we can observe the density of complete bid vectors $g(\mathbf{b})$ and we can identify $s(\mathbf{b})$. This is enough to identify the density of bid-types:

$$\widetilde{f}(\mathbf{b}) = k\frac{g(\mathbf{b})}{s(\mathbf{b})}$$

where the constant $k = \int s(\mathbf{b})\widetilde{f}(\mathbf{b})d\mathbf{b}$, the probability that a randomly chosen bidder's bid vector is complete, is observed in the data. Examining this expression, the selection correction "up-weights" the types who are least likely to bid on all products. Applying the inversion formula in (8) to $\tilde{f}$ gives us the density of valuations f and hence proves non-parametric identification in this two good example.

4.2 Identification Theorems

In this section we state and prove the main identification result of this paper. We work with the following demand system:

$$x_{i,j,l,t} = \check{x}_{i,j} + z_{l,t}\gamma + \xi_l + \varepsilon_{i,t} \tag{9}$$

where $x_{i,j,l,t}$ is bidder i's valuation for item l—an instance of good j—sold at time t, $\check{x}_{i,j}$ is bidder i's perfectly persistent valuation for good j (the vector $\check{\mathbf{x}}$ is sampled from $\mathbf{F}$), $z_{l,t}$ are (potentially time-varying) characteristics of this particular item, γ is a vector of common preferences for those time-varying characteristics, ξ_l is an item-specific specific shock sampled iid across items from F_ξ and $\varepsilon_{i,t}$ is an idiosyncratic shock sampled iid across bidders and auctions from F_ε. We assume that both ξ_l and $\varepsilon_{i,t}$ have zero mean and finite variance, and that they are mutually independent of each other and the observables $z_{l,t}$.

This demand system generalizes the example presented above in two ways. The first is that we include both item-specific observables ($z_{l,t}$) (e.g. this camera comes bundled with a battery) and unobservables (ξ_l) (e.g. this camera appears to be in good condition, from the photos). We assume that the observables $z_{l,t}$ are independent of the transient unobservables ξ_l and $\varepsilon_{i,t}$, though we place no restrictions on the correlation between $z_{l,t}$ and the persistent valuation $\check{x}_{i,j}$. The second is a mean-zero idiosyncratic shock $\varepsilon_{i,t}$. This shock allows us to rationalize data where we see variation in bids over time by the same bidder on the same item. We assume that the idiosyncratic shock is privately known by each bidder, and unobserved by the econometrician.

The demand system in (9) is quite flexible, placing limited restrictions on the data generating process. It can be made more or less flexible depending on how the goods indexed by j are defined. For example, defining all compact cameras to be a single good reduces the model to one in which consumers have common preferences up to a scalar error term. On the other hand, if we take every item with different observables as a distinct good (e.g. 7MP cameras with batteries are distinct from 7MP cameras without), the demand system becomes extremely flexible, as one can identify the joint distribution of valuations for each of these goods (and the term $z_{l,t}\gamma$ becomes redundant). The cost is that the cardinality of the product space becomes large, and the data requirements for estimation explode accordingly.

We can accommodate a demand system with random coefficients on product characteristics as a special case (see Corollary 1). In fixed price demand systems, an identification challenge is posed by correlation between the prices and the unobserved quality term (here ξ_l). Here this problem doesn't arise, because prices are an outcome of the auction, rather than a choice of an optimizing seller. That said, we have not allowed for reserve prices in the current paper, and if we had, it would be reasonable to assume they were correlated with ξ_l.[15]

We also assume that the observables $z_{l,t}$ are independent of the item level unobservables $\xi_{l,t}$. This is consistent with the IO literature on fixed price markets (product characteristics are assumed exogenous), but just as in that literature, it may be dispensed with if we assume the existence of appropriate instruments for the endogenous observable characteristics.[16]

[15] Existing approaches based on inverting reserve prices to recover unobserved quality may be of use in this case—see for example Roberts (2013) and Decarolis (2018).

[16] Below, we argue that the coefficient γ may be consistently estimated by an OLS regression of the bids

We can apply Lemma 2 to this demand system. Here the public state variables are the good being listed j and the commonly observed item shock ξ_l. The distribution of highest competing bids $G_j^1(\cdot|\xi)$ is conditioned on the realization ξ, and has additional variance relative to the model with permanent types because of the idiosyncratic shocks. We will assume throughout what follows that the distributions of highest bids $\{G_j^1(\cdot|\xi)\}$ have no gaps in their support, so that the best responses are unique. Then optimal bids are given by:

$$b_{i,j,l,t} = \check{x}_{i,j} - r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}}_i) + z_{l,t}\gamma + \xi_l + \varepsilon_{i,t} \tag{10}$$

where $v_k(\check{\mathbf{x}})$ is the perceived continuation value of a bidder of type $\check{\mathbf{x}}$ who is active and bidding on product k. Notice that the continuation value does not depend on the current shocks ξ_l and $\varepsilon_{i,t}$, because they are independently distributed over time and so today's shocks are not informative as to future payoffs.

We now turn to identification. We assume that in each auction, the econometrician observes the product-type being sold, all bids and the identities of the bidders that made them. We proceed with a sequence of lemmas.

Lemma 3 (**Identification of Dynamic Parameters**). *The parameters $\{\lambda, \tau, r, Q\}$ are non-parametrically identified.*

The logic follows that offered in the example above. Identification of Q is immediate, since it is directly observed. λ is identified from the mean number of entrants each period. Finally equations (5) and (6) still apply in the general case, implying identification of the activity and survival rates.

Next, we argue that the parameter γ can be identified in a first-stage (cf. Haile et al. (2006)). Define a bid-type on each product j by $\check{b}_{i,j} = \check{x}_{i,j} - r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}}_i)$. This is the *persistent* part of the optimal bid made by bidder i on product j. Then the bidding function in (10) can be written as:

$$b_{i,j,l,t} = z_{l,t}\gamma + \nu_j + \xi_l + \varepsilon_{i,t} + u_{i,j}$$

where $\nu_j \equiv \mathbb{E}_i[\check{b}_{i,j}]$ is the expected bid-type on product j, and $u_{i,j} = \check{b}_{i,j} - \mathbb{E}_i[\check{b}_{i,j}]$ is the difference between the bidder i's bid-type for product j and the average. Now by assumption

on the observable characteristics $z_{l,t}$, including fixed effects for each of the goods $j = 1 \dots J$. In the presence of endogenous characteristics and instruments, one could instead do an IV regression of the bids on the observables, instrumenting as necessary, and once again including product fixed effects.

$z_{l,t}$ is independent of ξ_l and $\varepsilon_{i,t}$. Moreover because bidders do not select into auctions on the basis of their characteristics $z_{l,t}$, the error $u_{i,j}$ is uncorrelated with $z_{l,t}$.[17] The only term potentially correlated with $z_{l,t}$ is ν_j, since certain products may tend to have certain characteristics. In view of this, γ is identified (and may be consistently estimated) by OLS regression of the bids on the observables, with product fixed effects.[18]

For the rest of this section we therefore assume that γ has been identified and all bids have been "normalized" by subtracting the term $z_{l,t}\gamma$ (i.e. we will simply ignore the term $z_{l,t}\gamma$ in the identification arguments that follow). Let the full bid type be $\check{\mathbf{b}}$ (i.e. the J vector consisting of $\check{b}_{i,j}$ for each $j = 1 \ldots J$). Also let $B^1_{j,l,t}$ be the winning bid in each auction, and define $\check{G}^1_j$ as the distribution of $\check{B}^1_{j,t} \equiv B^1_{j,l,t} - \xi_l$ (i.e. the distribution of the highest bidder-specific part of the bids). Let $u(\check{\mathbf{b}})$ be defined as follows:

$$u_j(\check{\mathbf{b}}) = \int \check{G}^1_j(\check{b}_j + \varepsilon) \left(\check{b}_j + \varepsilon - E_{\check{G}^1_j}[B^1_j | B^1_j < \check{b}_j + \varepsilon] \right) dF_\varepsilon(\varepsilon). \tag{11}$$

It turns out that $u(\check{\mathbf{b}})$ is exactly the "annuity payoff" that subsequently gets summed to derive a continuation value:

Lemma 4 (Inversion). *There exists an inverse bidding function $\zeta : \mathbb{R}^J \to \mathbb{R}^J$ mapping bid-types into persistent valuations:*

$$\zeta(\check{\mathbf{b}}) = \check{\mathbf{b}} + r\widetilde{Q}(I - r\widetilde{Q})^{-1} u(\check{\mathbf{b}}) \tag{12}$$

The argument is similar to the one in the example above: values are equal to bids plus continuation values, and continuation values are a discounted and summed geometric series of per period payoffs of $u(\check{\mathbf{b}})$. The problem is that the inverse bid function ζ is not as easily identified from the data, since from the final line of (11) the $u(\check{\mathbf{b}})$ vector can be constructed only with the knowledge of the $\check{G}^1_j$ bid distributions and the distribution $F_\varepsilon(\varepsilon)$.

To proceed, notice that G^1_j—the observed distribution of highest bids on good j—is a con-

[17]Even with selective entry there is likely to be no correlation: since all the bidders have common preferences for $z_{l,t}$, the bidders expect the value of those characteristics to be bid out, and so there is no strategic reason to enter auctions with particular values of the observables. This is true of our application, where even though we only observe bids from the top two bidders—and therefore the bid types in auctions for high camera resolution are positively selected for their preferences for high resolution—there is still no correlation between the other observable characteristics and bid type.

[18]In our empirical application, we use generalized least squares (GLS), which is more efficient for our particular specification. We thank the editor for this suggestion.

volution of the distribution we want ($\check{G}^1_j$) and F_ξ, the distribution of item shocks. If we can identify F_ξ, then under some technical conditions, we can also identify $\check{G}^1_j$ by deconvolution. To identify F_ξ, we look at the joint distribution of pairs of bids placed by bidders that have just entered the market. Since both the persistent part of their valuations and the idiosyncratic shocks $\varepsilon_{i,t}$ are mutually independent on entry, any correlation we observe in their bids must be due to unobserved heterogeneity.[19] This pins down F_ξ.[20] To identify F_ε we look instead at the joint distribution of pairs of bids of the same bidder in two successive auctions of the same item. This joint distribution of bids can be decomposed into a persistent part that induces all correlation, the sum of $\check{b}_j$ and ξ_l, and a transient part, $\varepsilon_{i,t}$, that is independent across auctions.[21] This allows us to identify $F_\varepsilon(\varepsilon)$.

Lemma 5 (**Deconvolution**). *Suppose (i) that the density of idiosyncratic shocks satisfies a thin tails condition: $f_\varepsilon(u) < c_1 e^{-c_2|u|}$ for large u, and (ii) that the characteristic functions of ε and ξ have isolated real zeros. Then $\{\check{G}^1_j\}_{j\in\mathcal{J}}$, F_ξ and F_ε are all identified, and thus so is $\xi(\check{\mathbf{b}})$.*

The thin tails condition (i) suffices to identify F_ε and F_ξ, by application of the results in Evdokimov and White (2012), who extend the deconvolution results of Kotlarski (1967). To subsequently recover $\{\check{G}^1_j\}_{j\in\mathcal{J}}$ requires deconvolution of the highest bid distributions $\{G^1_j\}_{j\in\mathcal{J}}$ with F_ξ, which is permitted by condition (ii).[22] Both conditions are weak and apply to most distributions commonly encountered including the normal, truncated normal and uniform.

The last step of the formal identification argument is arguing that the distribution of bid-types is identified, and thus can be inverted to recover the valuations. This result extends the selection correction arguments offered in the example to the general case of J-goods. As mentioned during the discussion of the example, it will considerably simplify the arguments if we assume that the probability of winning is independent of the bidder's experience i.e. that whether they win when bidding on good j today is independent of the fact that they lost on good k the last time they bid. To formalize this, let $h \in \mathcal{H}$ be a bidder history: a set

[19] Conditioning on having just entered is important, since selection on having repeatedly lost may be another source of correlation.

[20] This is similar to the argument made in Krasnokutskaya (2011) regarding the identification of unobserved heterogeneity in procurement auctions.

[21] The transient part has a different distribution in the second auction than the first, since appearing in the second auction implies selection on the idiosyncratic error in the first - we account for this in the proof.

[22] Let $\phi_X(\cdot)$ denote the characteristic function of a random variable X. By independence, we have $\phi_{\check{B}^1_j}(t) = \phi_{B^1_j(t)}/\phi_\xi(t)$, which is undefined whenever $\phi_\xi(t) = 0$. Still $\phi_{\check{B}^1_j}(t)$ remains integrable as long as the set of such zeros is of (Lebesgue) measure zero, so that $\check{G}^1_j$ is identified by inverse Fourier transform.

of tuples, one tuple for each time period in which the bidder was active, where each tuple consists of the time period they were active in, what good they bid on in that time period, what they bid, and whether they won or lost (e.g. "bid b on good 1 in period 12, lost").

Assumption 3 (Independence of competing bids). *The distribution of highest rival bids is independent of bidder history i.e.* $G_j^1(b|h) = G_j^1(b)\ \forall j\ \forall h \in \mathcal{H}$.

This is an empirical counterpart to Assumptions 1 and 2. There we insist that bidders don't condition on their own personal experience in forming beliefs about rivals; here we assume that doing so would not be informative.

Implicitly this is an assumption that the activity rate τ is small, since the market quickly converges back to the ergodic measure from any state (see Lemma 6 in the appendix for a formal statement and proof of this), so that if the number of auctions between each time a bidder participates is large, this assumption is approximately correct.[23]

What this assumption buys us is the Markov state formulation of the selection problem from the example. Define $s(\check{\mathbf{b}})$ to be the probability that bid-type $\check{\mathbf{b}}$ bids at least once on every product prior to exiting the market. In this general case, the probability of survival following an auction is slightly different due to the presence of shocks: $r_j(\check{\mathbf{b}}) = r\left(1 - \int \check{G}_j^1(\check{b}_j + \varepsilon)dF_\varepsilon(\varepsilon)\right)$, where $\int \check{G}_j^1(\check{b}_j + \varepsilon)dF_\varepsilon(\varepsilon)$ is the ex-ante probability of winning auction j, leaning on Assumption 3 to avoid conditioning on the bidder's record. As in the example, the probability of visiting all transitory states prior to exit depends only on the transition matrix, which in addition to $r_j(\mathbf{b})$ depends on $\widetilde{Q}$. From Lemma 3 and Lemma 5 these objects are both identified, and thus so is $s(\check{\mathbf{b}})$ (for a more detailed proof see Lemma 8 in the appendix).[24]

The last stage of our identification argument again makes use of complete bid vectors. In contrast to the example, in this demand system a bidder may have multiple distinct bids on the same good, due to shocks. For each bidder who bids on every product, define a bid vector $\mathbf{b}$ by selecting from their bid data the bid corresponding to the first time they bid on each product, with missing entries for any product they did not bid on. A complete bid

[23] In our empirical application, if we assume that all bids are observed, then on average 305 auctions pass between bids by the same bidder, implying $\tau = 0.00327$.

[24] We note that since the conditional bid distributions $G_j^1(b|r)$ are observable, the identification arguments made here can be extended to do without Assumption 3, at the cost of some additional complexity. The details are available in an earlier draft of this paper, and are available on request.

vector has no missing entries, and from equation (10) can be written as:

$$\mathbf{b} = \check{\mathbf{b}} + \boldsymbol{\xi} + \boldsymbol{\varepsilon} \tag{13}$$

where $\check{\mathbf{b}}$ is their bid-type, $\boldsymbol{\xi}$ is the J-vector of item-specific shocks in the auctions they participated in, and $\boldsymbol{\varepsilon}$ are the J-vector of idiosyncratic shocks. Each of the random variables on the RHS is mutually independent, and the LHS is a convolution of these random variables.

This suggests an identification argument based on deconvolution. Let $\mathbf{G}$ be the ergodic distribution of complete bid vectors, and let $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{F}}^S$ be the distribution of bid-types, and the distribution of bid-types conditional on bidding at least once on every good respectively. The relationship between $\widetilde{\mathbf{F}}^S$ and $\mathbf{G}$ is given by:

$$\mathbf{G} = \widetilde{\mathbf{F}}^S * \mathbf{F}_\xi * \mathbf{F}_\varepsilon \tag{14}$$

where $*$ denotes the convolution operator, $\mathbf{F}_\xi$ is the distribution of $\boldsymbol{\xi}$ and similarly $\mathbf{F}_\varepsilon$ is the distribution of $\boldsymbol{\varepsilon}$. Since $\mathbf{G}$ is a distribution of bid vectors, it is observable. By Lemma 5, the univariate F_ε and F_ξ are both identified and have isolated real zeros, and hence so do their vector counterparts $\mathbf{F}_\xi$ and $\mathbf{F}_\varepsilon$, which are just iid samples of size J from those distributions. This implies that $\widetilde{\mathbf{F}}^S$ can be identified by deconvolution from (14). Finally, we can re-weight $\widetilde{\mathbf{F}}^S$ according to the selection probability to get $\widetilde{\mathbf{F}}$ and thus $\mathbf{F}$ by Lemma 4. Summarizing:

Theorem 2 (**Non-parametric identification**). *Let Assumption 3 and the conditions of Lemma 5 hold. Then* $\mathbf{F}$*,* F_ξ *and* F_ε *are all non-parametrically identified.*

We close the section by examining a slight modification: a random coefficients demand system, where goods are described by observable characteristics and bidders have preferences over characteristics rather than goods.

$$x_{i,j,l,t} = W_j \boldsymbol{\alpha}_i + z_{l,t}\gamma + \xi_l + \varepsilon_{i,t} \tag{15}$$

where now W_j is the jth row of a $J \times K$ matrix W of product characteristics (where $K \leq J$ is the number of characteristics) and $\boldsymbol{\alpha}_i$ is the bidder's type, a persistent vector of random coefficients drawn from $\mathbf{F}_\alpha$ on entry (the definitions of all other expressions remain the same).

This is very similar to the demand system in (9) except that the persistent valuations for each product have been replaced with persistent preferences (random coefficients) over a lower dimensional set of product characteristics W (which are different from the item characteristics $z_{l,t}$). This suggests that we could go ahead and learn the distribution of valuations $\mathbf{F}$ and then project down onto W to learn the distribution of random coefficients $\mathbf{F}_\alpha$. Under a rank condition on W, this is possible:

Corollary 1 (**Identification with Random Coefficients**). *Let Assumption 3 and the conditions of Lemma 5 hold. In addition, assume that the design matrix W has full rank. Then $\mathbf{F}_\alpha$, F_ξ and F_ε are all non-parametrically identified.*

5 Empirical Application

We apply our identification result to estimate demand in the auction market for compact cameras on eBay.com, using a specification of the form in (15) above. As noted earlier, the eBay auction design is strategically similar to a sequence of second-price sealed bid auctions. Moreover, most consumers only purchase a single camera, so the unit demand assumption seems reasonable.

Compact cameras are measurably differentiated in a salient characteristic—namely resolution—which consumers may value differently, and therefore we estimate a random coefficients variation of our model. The demand model we estimate allows for random consumer preferences for resolution. We estimate the parameters of this model using a maximum likelihood approach.

The application is motivated by our ambition to show how one might bridge the gap from identification to estimation in a particularly difficult setting with data limitations, unobserved heterogeneity, and random preferences; i.e., to develop an estimator that takes advantage of much of the flexibility of the identification results. As in all empirical work, this flexibility imposes tradeoffs. Here we have taken the supply process that determines which camera is auctioned next as exogenous, and in addition we have assumed that sellers do not use reserve prices strategically (a feature that is easy to verify in the data). These are limitations of our approach.

Nonetheless, we put our estimates to use by documenting the effects of dynamic bidder behavior on optimal reserve prices. As we show, if sellers have the power to commit to persistent reserves over a long time horizon, this diminishes the bidders' option value of losing, which means higher bids. In our simulations, this effect can as much as double expected seller revenue vis-á-vis comparable transient reserve prices, offering insight on how market power might manifest in an auction marketplace with large sellers.

5.1 Data and Market Overview

The eBay Marketplace. eBay is widely considered to be the world leader in online auctions. Various elements of its platform design, such as the use of proxy bidding agents, feedback scores and "buy-it-now" offers have been widely copied. At any time, eBay hosts a large number of items from a variety of sellers. Buyers can browse these, either by navigating through categories delineated by the site, or by directly searching for key phrases. For example, a search for "digital compact camera" will typically bring up thousands of listings. Unsurprisingly, there is substantial heterogeneity in the cameras offered, in terms of brand, resolution, zoom and accessories (to name some of the most salient features).

Restricting to auctions yields a list of items, ordered by time until auction end.[25] These auctions all end at different times, so bidders face a set of *sequential* auctions. Bidders have the options to bid using eBay's proprietary "proxy bidding" system, however much prior work has found that most bids are placed in the last day of the auction (Roth and Ockenfels, 2002; Lewis, 2007; Backus et al., 2015).[26] The combination of late and proxy bidding suggest that eBay's auction market is well approximated as a sequence of second-price sealed bid auctions, so our model can be applied to this setting.

Data. We purchased a dataset concerning all sales of digital compact cameras over a 2-year period from TeraPeak, a data analytics company. The data includes attributes of the camera

[25] Some items are offered at "buy-it-now," i.e. fixed, prices. As Einav et al. (2016) have documented, over 60% of items on eBay are now at fixed prices (though auctions were more common at the time our data was collected). Backus et al. (2019) and Backus et al. (2020) study a subset of these items are "best offer"-enabled, and therefore subject to alternating sequential offers bargaining. We will ignore the presence of both fixed-price and negotiated markets in what follows, effectively ignoring substitution opportunities between the two markets in favor of focusing on substitution within the auction market.

[26] In the proxy bidding system, bidders enter the maximum they are willing to pay for the item, and then eBay's proxy bidding system will bid up from the current standing price in standardized increments on their behalf until either their bid is the highest yet entered in the system, or an additional increment would take them over their maximum. For example, if bidder A enters a bid of \$8000 on a camera where the standing price is \$6000 and the highest bid placed in the system by a rival is \$7000, then the system will update the standing price to \$7100 (\$7000 + \$100 increment), and will record this bidder as the currently high bidder. Under unit demand high bidders become "committed" to the auctions they enter, in the sense that if they bid in another auction, there is a risk of winning a second object they don't need.

auctioned (resolution, zoom, brand, product name, bundling of a tripod, extra battery etc), attributes of the listing (starting price, secret reserve, listing title), and the outcome of the auction. Each listing may be associated with several bids, all of which we observe—including the highest bid, which is not visible on the website and typically unavailable in "scraped" auction datasets from the platform. Market participants are persistent in our dataset—as in our model—and we observe their attributes (feedback, location) and construct measures of experience and activity from observed behavior.

We work with a subset of the data, consisting only of *new* compact cameras, sold in the 3-month period between February 5th and May 6th of 2007. We restrict attention to new cameras to limit the influence of unobserved heterogeneity, though as we will see, this is still a substantial problem. We analyze this particular time period because supply was relatively stable over those 3 months, so the stationarity assumptions implicit in our calculations of the continuation values are reasonable. We pick cameras with the most common resolution levels: those with (rounded) resolution between 5 megapixels and 10 megapixels (MP). We clean the data by excluding auctions with missing data, potential shill bidding, outlying bids and auctions that are terminated by the first bidder exercising a buy-it-now option (this is unusual). See the data appendix for further details on sample construction, as well as summary statistics of the dataset.

We follow Haile and Tamer (2003) and Song (2004) and restrict attention to the first- and second-highest bids in each auction. This is motivated by a disconnect between the sealed-bid abstraction and the actual, ascending-price character of online auctions. For bidders who are outside of the top two, we may not observe the highest amount they would have been willing to bid (for instance, if they arrive after the standing price of the auction already exceeds the bid they intended to place).

A first look at the dataset yields four stylized facts that guide the design of the empirical model in what follows. We document each of these claims more carefully in Appendix Section B.3. First, bidders do substitute across auctions for different camera types. This is important for identifying cross-elasticities. Second, consistent with our theoretical model, the vast majority of sellers set non-binding starting prices and do not use secret reserve prices. This is demonstrably suboptimal, but setting an optimal reserve is a difficult problem in practice, and the platform encourages sellers to set low starting prices.[27] Third, bidders have substantial option value. We observe substantial price fluctuations over time, as well

[27]See, e.g., https://www.ebay.com/help/selling/selling/pricing-items?id=4133. Accessed July 23, 2020.

as substantial delay in bidder re-arrivals. Fourth and finally, there is substantial unobserved product heterogeneity, as the observables explain little of the fluctuations. Therefore we endeavor a model that can accommodate these salient features of the data.

5.2 Estimation

We construct a demand system in which consumers obtain value from the purchase of a single camera, a value which has an idiosyncratic component and a common component. Formally, bidder i's valuation for product j offered in auction t is given by:

$$x_{i,j,l,t} = \underbrace{\alpha_{i,c} + \text{res}_j \alpha_{i,r}}_{\text{idiosyncratic}} + \underbrace{Z_{j,t}\gamma + \xi_l}_{\text{common}} . \tag{16}$$

This combines random coefficients and unobserved heterogeneity to exploit the power of Corollary 1 and adds some observable common demand shifters $Z_{j,t}$ that will be estimated and subtracted out in a first stage. Bidder i's type in our model, α_i, is a double: their fixed utility draw for obtaining any camera ($\alpha_{i,c}$) as well as an idiosyncratic preference shock for resolution ($\alpha_{i,r}$).[28] The auction-specific term ξ_l captures unobserved heterogeneity that is observable to all bidders, but not the econometrician. We assume that it is distributed normally with mean zero and variance $\sigma_{\xi,j}$ that varies freely with the resolution type of the camera. The bidder type α_i is drawn, iid upon bidder entry, from the distribution $\mathbf{F}_\alpha$, which is our main estimation target. We assume mutual independence of $(\alpha_i, Z_{j,t}, \xi_l)$. On the supply side, we make the simplifying assumption that the distribution of arriving auctions is iid multinomial over the product space with a J-vector of probability weights π (note that this implies that Q is a square matrix in which every row is equal to π).[29][30]

[28]An alternative strategy would have been to allow for random coefficients in bidders' preferences for each brand. But because the set of brands is large, this would introduce many additional parameters. We follow the fixed-price demand estimation literature in focusing on random preferences for low-dimensional characteristics.

[29]We have made these choices to fit the salient features of the data, however we acknowledge there are other models that may be of interest, depending on the application. See Appendix Section B.4 for a fuller discussion on this point. In particular, we consider the incorporation of aggregate fluctuations and public states.

[30]Note also that we do not include a transitory idiosyncratic shock (ε_{it}), which implies that bidders bid the same amount on the same product, up to unobserved heterogeneity (ξ_l). Therefore, if two bidders are observed bidding on the same product, the difference between their bids ought to be constant (thanks to an anonymous referee for pointing this out). Due to our restriction to first- and second-highest bids, this scenario never occurs, but if it did then the additional flexibility afforded by a transient shock would be

First-Stage Estimation. We follow the constructive identification argument of Lemma 3 to recover estimates of the supply process (π), the exit rate (r) , the desired bid normalization (γ), as well as the distribution of observed bids ($G^{(1)}$ and $G^{(2)}$). $\hat{\pi}$ is given by the empirical frequencies; $\hat{r}$ is estimated using a negative binomial process predicting exit rates among the sample of bidders who never win, corrected for censoring at the end of our dataset; $\hat{\gamma}$ is obtained by FGLS,[31] $\hat{\sigma}_\xi$ is the standard deviation of within bidder-product differences in bids, and finally, the shape and scale parameters of $\hat{G}^{(1)}$ and $\hat{G}^{(2)}$ are chosen to match the mean and variance of the deconvoluted distribution. As controls in the first-stage regressions we include product line fixed effects, listing attributes including shipping options, seller feedback, and optional listing features (e.g., sellers may pay a fee for their results to be highlighted in search results), as well as a set of dummies for resolution, optical zoom, and digital zoom levels. See Appendix Section B.5 for further details, as well as precise values in Table B-4. Also, at this point it is possible to estimate consumer surplus in each auction, which is substantially larger than would be predicted by the static model. We discuss this in Appendix Section B.8.

Estimation of Demand. At this stage, we have estimates of π, r and the distributions $\{G_j^1, G_j^2\}$ of first and second order statistics, after adjusting for observed and unobserved heterogeneity. We also have a dataset of normalized bids that were themselves first or second highest bids in auctions, and therefore plausibly equal to valuation less continuation value. In principle, we could continue by applying our nonparametric identification strategy from Theorem 2 directly in estimation, for bidders who are observed in the top two bids on at least two different products (since the random coefficient is two-dimensional, two different products suffices for identification). But there are 232 such bidders in our estimation sample, and this makes the necessary deconvolution analysis unattractive in view of the slow convergence properties of such estimators (Carroll and Hall, 2004). Moreover, we would like to also use the 13,392 bidders in our estimation sample who appear in the top two bids on only one camera type.

necessary to fit the data.

[31]There are two important things to note in the first-stage regressions. First, the constant and any coefficient on resolution would be biased due to non-random selection on α_c and α_r; therefore we include dummy variables for resolution types and add the estimated values back in when computing normalized bids. Second, and thanks to editor Aureo de Paula for this suggestion, we use FGLS, allowing the error to be correlated with resolution, because idiosyncratic types α_r introduce heteroskedacticity, rendering OLS (which we used in a prior version) inefficient.

So instead we take a parametric approach, assuming that α_c and α_r are distributed $N(\mu_c, \sigma_c)$ and $N(\mu_r, \sigma_r)$, respectively. Therefore $\theta \equiv \{\mu_c, \sigma_c, \mu_r, \sigma_r\}$ makes up the set of parameters of the demand system we ultimately hope to recover. Under our parametric assumptions, we have the following likelihood of each observation:

$$\mathcal{L}(\mathbf{b}_i|\theta) = \int \underbrace{\frac{\mathbb{P}\{\mathcal{B}_i|\beta(\boldsymbol{\alpha})\}}{\mathbb{P}\{|\mathcal{B}_i| = 2|\beta(\boldsymbol{\alpha})\}}}_{\text{selection probability}} \left(\underbrace{\prod_{j \in \{\mathcal{B}_i\}} f_{\xi,j}(b_j - \beta_j(\boldsymbol{\alpha}))}_{\text{unobserved heterogeneity}} \right) |\beta'(\boldsymbol{\alpha})| d \underbrace{F(\boldsymbol{\alpha}|\theta)}_{\text{demand}}. \tag{17}$$

where $\mathbf{b}_i$ is a two-vector, indicating an observed pair of bids by i; $\mathcal{B}_i$ is the set of products bidder i bids on; $\beta(\boldsymbol{\alpha})$ is type $\boldsymbol{\alpha}$'s bid-type (i.e. what they would bid on each product in the absence of unobserved heterogeneity); $f_{\xi,j}$ is the density of ξ_l, assumed normal with mean zero and variance $\sigma^2_{\xi,j}$ and $F(\boldsymbol{\alpha}|\theta)$ is the distribution of $\boldsymbol{\alpha}$ at parameter vector θ.

This likelihood function has three components in addition to the Jacobian:[32] the first, $\mathbb{P}\{\mathcal{B}_i|\beta(\boldsymbol{\alpha})\}/\mathbb{P}\{|\mathcal{B}_i| = 2|\beta(\boldsymbol{\alpha})\}$ is the selection probability; the likelihood that a bidder of type $\boldsymbol{\alpha}$ is observed in a subset of the product space $\mathcal{B}_i$ conditional on the event $|\mathcal{B}_i| = 2$, i.e. our sample construction. The second component of the likelihood function is the deviation of the observed bid $\mathbf{b}$ from the predicted bid $\beta(\boldsymbol{\alpha})$ on the components $j \in \mathcal{B}_i$, which can be accounted for by unobserved heterogeneity. Finally, we integrate with respect to the type distribution $F(\alpha|\theta)$, the only point at which the parameter vector θ enters.

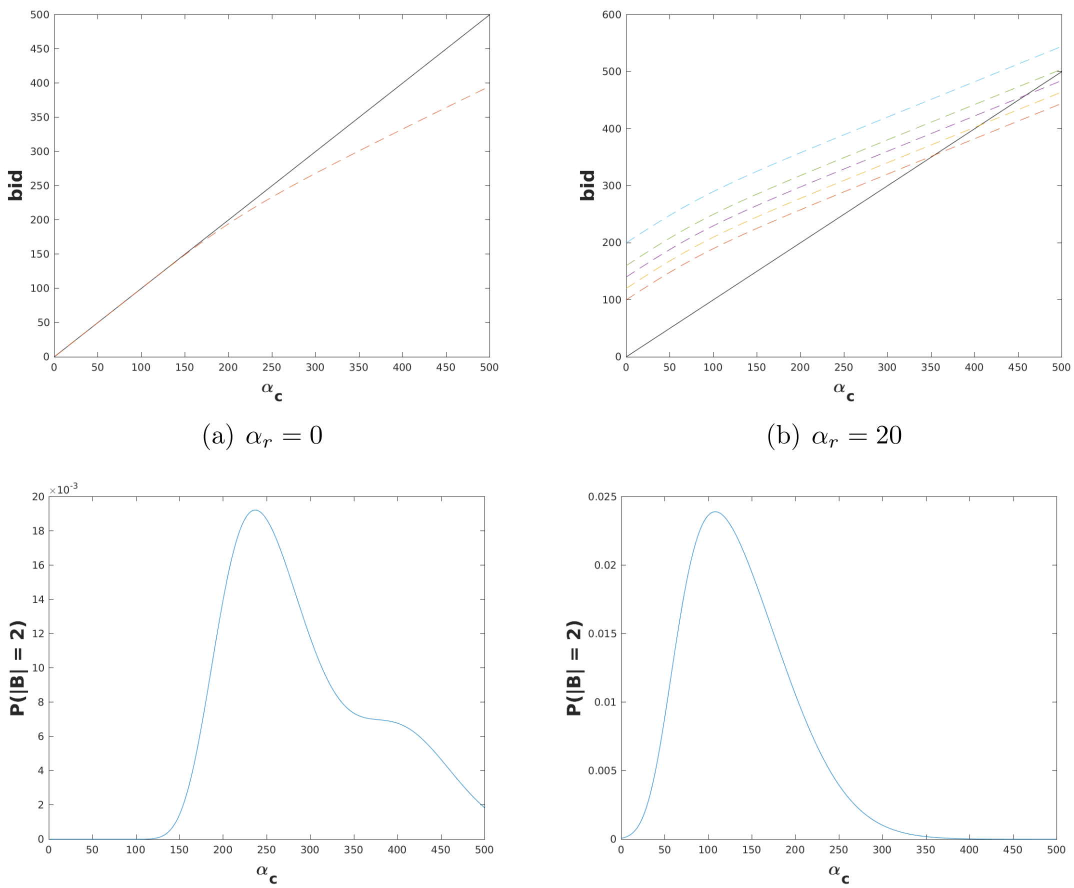
Estimating bid functions and selection probabilities. In order to compute the likelihood of any observation at a parameter vector θ, we will need to compute the optimal bidding function $\beta(\boldsymbol{\alpha})$ and the selection probability $\mathbb{P}\{\mathcal{B}_i|\beta(\boldsymbol{\alpha})\}$. See Appendix B.6 for details on this computation, which follows closely the identification arguments of Section 4.

Figure 2 illustrates the resulting bid functions and selection probabilities when these proce-

[32] The Jacobian of interest here sits in a lower-dimensional space than the bid vector. Here, we construct it as

$$|\beta'(\boldsymbol{\alpha})| = |\frac{\partial \beta_i}{\partial \boldsymbol{\alpha}_c}\frac{\partial \beta_{i'}}{\partial \boldsymbol{\alpha}_r} - \frac{\partial \beta_{i'}}{\partial \boldsymbol{\alpha}_c}\frac{\partial \beta_i}{\partial \boldsymbol{\alpha}_r}|,$$

– where i and i' are the two resolution types observed for that particular bid. However, since we only need two elements to invert back to $\boldsymbol{\alpha}$, it is possible to show that the likelihood function is invariant to the choice of i and i'.



(a) $\alpha_r = 0$ (b) $\alpha_r = 20$

(c) $\alpha_r = 0$ (d) $\alpha_r = 20$

Figure 2: Estimated Bid Functions and Selection Probabilities. This figure presents bid functions and selection probabilities as a function of α_c, holding fixed α_r at two levels: 0 and 20.

dures are applied to our data. The selection probability is for the event that a bidder bids on exactly two different products ($|\mathcal{B}| = 2$), i.e. that a bidder is selected into our estimation sample. Panel (a) shows the bid function for a bidder with $\alpha_r = 0$, i.e. a bidder who places no value on resolution, and consequently bids the same on every product. Their bid function "peels away" from the 45-degree line. This makes intuitive sense: as α_c increases, their continuation value increases as well, causing them to shade their bids. Panel (b) repeats this exercise for the case where $\alpha_r = 20$. They now bid differently on each product, with the five dotted lines showing their bids on the lowest resolution camera (bottom line) up to the top resolution camera (top line), as α_c varies. The shape is largely preserved, except that now there is a positive intercept because valuations are positive even with $\alpha_c = 0$.

In panels (c) and (d) we show the selection probabilities for these two types of bidders ($\alpha_r = 0$

in (c) and $\alpha_r = 20$ in (d)) as α_c varies. The "hump-shape" arises because the probability of $|\mathcal{B}| = 2$ is increasing at first as the likelihood that the bidder is ever first or second rises, but later declines as the probability that they win their first auction and exit before bidding again goes to one.

Optimization. See Appendix B.7 for a detailed discussion of how we solve the ML optimization problem. We show that, because the parameter vector θ enters the likelihood (17) only through the distribution of random coefficients $F(\alpha|\theta)$, this problem is particularly well-suited to importance sampling (Kloek and van Dijk, 1978), which yields dramatic computational gains in estimation.[33]

5.3 Results

Estimates from this ML exercise are presented in Table 1.[34] Our results suggest that camera resolution is the main determinant of consumer utility, but that there is substantial heterogeneity in how consumers value this attribute. This variation in the two components of utility for a camera predicts that the winner of an auction will depend not merely on the valuations of the bidders but also which camera is up for auction. A bidder with a high α_c and low α_r will bid more aggressively in an auction for a 5MP camera, while a bidder with a low α_c and high α_r will bid more aggressively in an auction for a 10MP camera. Recall that we have estimated $\mathbf{F}$, whereas the observed bids in the data (i.e., first- or second-highest bids) will tend to come from bidders sampled the right tail of the distribution.

5.4 Counterfactual Reserve Prices

With our estimates of the demand system in hand, we consider the problem of a large seller with market power, who controls approximately a third of the market, and can set reserve prices in these auctions. We are interested in this scenario because, for lack of models that allow substitution between auctions, little is understood about how market power manifests

[33]The argument for importance sampling as a technique for reducing the computational burden of structural estimation is due to Ackerberg (2009).

[34]The ML estimator yields standard errors that treats the first-stage parameters as known, and will therefore be too small (Murphy and Topel, 2002; Karaca-Mandic and Train, 2003). Following the suggestion of Petrin and Train (2003), we augment these standard errors by bootstrapping the first-stage estimates, clustered by bidder, and re-estimating the model for each bootstrap.

Table 1: Demand System Estimates

	α_c	α_r
Mean	1.6664	34.4463
	(1.7798)	(0.2747)
Standard Deviation	35.3059	5.9212
	(1.0554)	(0.1460)

Notes: This table presents estimates for $\{\mu_c, \sigma_c, \mu_r, \sigma_r\}$, the parameters governing the distribution of random coefficient preferences for compact cameras according to equation (16).

in auction marketplaces.[35] For convenience of computation, we assume that this seller sells all of the 7 megapixel cameras. As a baseline, we first simulate the expected sales prices for each kind of good, assuming no reserves. The seller then sets reserves that are parameterized by a single parameter f, the fraction of the baseline expected sales price. Throughout, we assume that sellers of other cameras (5, 6, 8, and 10 megapixel cameras) do not use reserve prices.

We consider two types of reserve pricing policies: a *transient* reserve, where the monopolist sets a reserve price in the current auction alone, and a *persistent* reserve policy, where the seller commits to set reserves in the current auction and all future ones. A transient reserve price may raise revenue by increasing the expected payment when it binds (at the cost of fewer sales when no bid exceeds the reserve). A persistent reserve price policy has the same effect, but in addition, it will also depress bidders' continuation values by raising expected prices in the future, which in turn pushes out the bid function and raises revenues still more.[36] It is the market power of the large seller that gives them the ability to shift bidder expectations, and therefore benefit more from a reserve policy. In this sense, our counterfactual offers a first step towards understanding the exercise of market power in a dynamic auction marketplace.

In order to compute this counterfactual, we simulate the market, fixing an entry process of $E = 2 + X$, where X is Poisson with $\lambda = 1.9$, so that the mean of 3.9 matches the empirical ratio of observed bidders to auctions in our dataset, and there are always at least

[35]Notable exceptions are McAfee (1993) and Peters and Serevinov (1997), however they focus on the competitive case, predicting that reserve prices in equilibrium will be equal to sellers' costs.

[36]There is a second-order effect as well: the persistent reserve price policy will change the ergodic distribution of bidders—on average there will be more bidders per auction, with more mass just beneath the reserve point cutoff. This, however, will not raise revenues for that particular product (although it may for others).

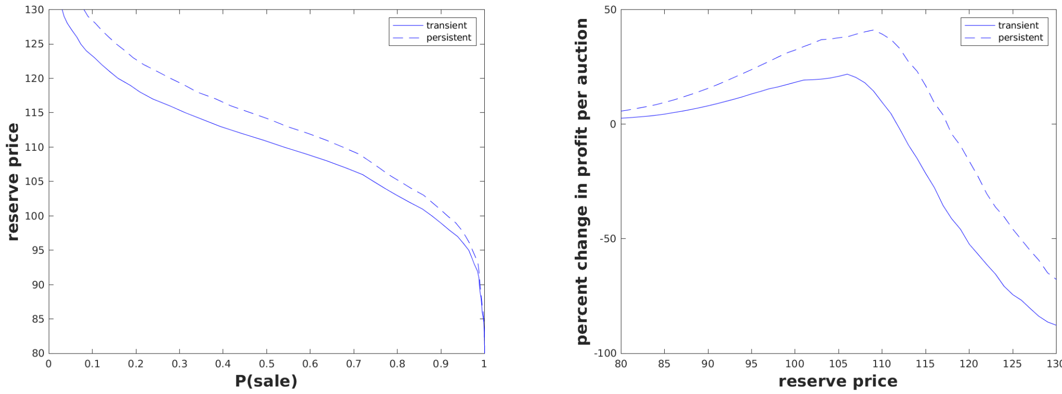


Figure 3: Demand and Profit Per Auction for 7 Megapixel Cameras. Panel (a) of this figure presents simulated demand functions for 7 megapixel cameras under transient (solid line) and persistent (dashed line) reserve price policies using the estimates of Table 1. Panel (b) depicts the corresponding profit function under an assumption that the seller's costs are equal to 90% of the baseline price.

two bidders.[37] Then we simulate outcomes under different reserve price policies for the seller. For the persistent case, this requires computing a fixed point between the bidding function and the distribution of the highest rival bids, which we compute by iteration: we compute the bid function, simulate auctions, estimate the parameters of the distribution of the highest competing bids, and repeat this process until it converges.

The effects of the persistent reserve policy are shown in Panel (a) of Figure 3. Here we plot the probability of sale against the reserve price for both transient and persistent reserve price policies, and we see that the persistent reserve price policy has rotated the demand curve outwards, particularly for higher reserve prices that are more likely to be pivotal. This rotation of the demand curve implies that the seller will have an incentive to set higher reserve prices, and will achieve higher profits. In Panel (b) of Figure 3 we plot the corresponding profits for different reserves under an assumption that the seller's costs are equal to 90% of the baseline sale price. These costs may be realistic if they are thought of as including the outside option to wait and sell in a subsequent auction, or sell the product on another platform. We see that profits are consistently higher under the persistent reserve price policy. Note that the profits achieved by a transient reserve are available to any seller, regardless of size. But it is only by virtue of the large market share of the seller that, through a persistent reserve price, they are able to depress bidders' continuation values, raise bids, and thereby achieve higher profits. The profit difference is the market power of the large seller.

[37]We do this because we almost never observe fewer than two bidders in the data.

Table 2: Optimal Reserve Prices

	Seller Costs as a Percent of ASP									
	0	10	20	30	40	50	60	70	80	90
Transient										
Optimal Reserve	83	83	83	83	83	91	91	92	95	106
Pct. Change in Profits	0.30	0.34	0.39	0.45	0.54	0.76	1.27	2.22	4.79	21.83
Pct. Change in Cons. Surplus	-1.60	-1.60	-1.60	-1.60	-1.60	-4.09	-4.09	-4.62	-6.77	-30.10
Persistent										
Optimal Reserve	84	93	93	93	93	93	93	97	99	109
Pct. Change in Profits	0.81	0.95	1.25	1.65	2.17	2.90	3.99	6.26	12.19	41.11
Pct. Change in Cons. Surplus	-3.91	-8.13	-8.13	-8.13	-8.13	-8.13	-8.13	-9.80	-10.58	-15.79

Notes: This table presents optimal reserves for the large (all 7mp cameras) seller, under transient and persistent reserve price policies, as well as percentage changes in profits induced by those policies, relative to the no-reserve baseline.

In Table 2 we consider the optimal reserve price problem of the seller for a range of seller costs. Using grid search over our simulated outcomes, we identified optimal reserves under both the transient reserve price policy as well as the persistent reserve price policy. We find that for low costs, the use of persistent reserves is of little consequence for the seller. However, for high costs, persistent reserves can have a substantial effect on seller profits, raising them by up to 41% over the no-reserve baseline in the range that we consider. The percentage change in profits for persistent reserves over the range is consistently double to triple the size of that predicted for transient reserves.

This highlights the importance of dynamics for understanding reserve prices, and the leverage that a large seller in an auction marketplace possesses. Although the level of the optimal reserve changes only a little, the incentives to implement it are dramatically larger when the seller can commit to future reserves, eroding the buyers' continuation values, and therefore raising their optimal bids. This is, to the best of our knowledge, a novel observation in the literature on reserve prices.[38]

In the bottom row for each case we also consider the effect on consumer surplus. On a per-auction basis, the transient reserve decreases consumer surplus by up to 15.79%, ap-

[38]Hickman et al. (2017), who like us find little use of reserve prices among eBay sellers, argue that the reason for this is that the gains from optimal reserves are quite small—on the order of pennies, in their application to laptops on eBay. While we find larger gains to transient reserves, on the order of dollars, we view our result as qualitatively consistent. However we also find that the gains are much larger for sellers with market power, which is consistent with casual empiricism: reserve prices tend to be used by long-run players with market power, e.g. procurement agencies and advertising platforms.

proximately half of the loss from the use of persistent reserve prices.This divergence is due to the fact that when reserve prices are persistent, bidders adjust optimally. The loss in surplus from reserve prices occurs when a listing does not sell even though the bidders' valuation is higher than the reserve (due to optimal shading); by raising their bids, bidders mitigate these losses.

A limitation of our analysis is that we take seller participation as exogenous. This is consistent with our focus on demand, however a fuller model that captured both sides of the market might incorporate the possibility that sellers who fail to sell—because they have set a reserve price—re-enter the market, thereby endogenizing their outside option. We have assumed instead that sellers obtain their (fixed) marginal cost (which may stand in for the outside option of selling in another market), in order to keep the seller side of the problem static. Nonetheless, our counterfactual exercise shows how explicitly modeling substitution between auctions—here, through the option value of waiting—allows us the opportunity to study the exercise of market power in auction marketplaces. Especially as these marketplaces proliferate, we believe this is an important direction for future research.

6 Conclusion

This paper offers a flexible demand system for the study of auction markets *as markets*. We developed a notion of equilibrium in such markets and proved its existence, and in turn were able to characterize the conditions under which bidders' actions can be inverted to infer their private type. While this is sufficient for identification if we treat auctions as independent, isolated draws from a distribution, in an auction marketplace we also need to account for the selection of bidders into the observed and identified set. Subject to the constraints of non-participation, we are able to partially identify the distribution of types by explicitly modeling this selection as a function of observable equilibrium objects. This selection correction turns out to be an important source of flexibility in the model, allowing us to accommodate standard limitations of auction data, such as only observing the two highest bids.

A second important source of flexibility in the model is that we have made all the substitution between products occur *inter-temporally*. High-dimensional preferences are thus projected down to a valuation for the current product and a continuation value which, thanks to the Markov dynamics of the game, we can model in a straightforward way. These Markov dynamics also give us the flexibility to extend the state space of the game to incorporate arbitrary public signals about the state of the market on which bidder behavior may depend. Of course, this comes at a cost, in that it restricts the way we think about bidder search and entry. A fruitful direction for future work, given a compelling model of consumer search, would be to use those search and entry choices to learn more about bidder preferences.

We illustrated much of this flexibility in an application to the auction market for compact cameras on eBay. There we estimated the distribution of types for a utility function that looked a lot like the kind you might estimate in a fixed-price market using standard methods in industrial organization. We applied the estimates to simulate a counterfactual world in which a large seller, controlling approximately one third of the market, could set reserve prices (expressed as a fraction of the baseline expected sales price) for all products, thereby depressing continuation values and raising current bids. We found that this effect can be quite large. It rotates the demand curve outwards and raises profits; for example, if the seller's costs are 90% of the baseline, then the profits of the seller are 41% higher under optimal persistent reserves rather than no reserve, versus 22% under an optimal transient reserve. This difference stems from the market power of the large seller.

Our contribution—the development of a demand system for auction markets—is meant to mirror similar work in fixed-price markets, and to a similar end: the structural estimation of demand allows us to do counterfactuals of both private and public interest. There remain several open directions for future work. This framework could potentially naturally to multi-unit demand, in which bidders may shade against the opportunity cost of moving further down their marginal utility curve. This is an important direction for the modeling of advertising auctions and treasury auctions. There are also unmet challenges in the modeling of substitution across auctions within-period rather than inter-temporally. Auction markets are a pervasive mechanism for the allocation of goods and services, and there remains much work to be done to understand competition between and substitution among them.

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Appendices

A Proofs

Proof of Lemma 1.

Proof. Recall that G_j^1 is the CDF of the highest competing bid for good j. Then we have:

$$\beta_j(\mathbf{x}) = \arg\max_{b \in \mathbb{R}} \int \big(1(B_j^1 \leq b)(x_j - B_j^1) + 1(B_j^1 > b)\tilde{v}_j(\mathbf{x})\big) dG_j^1(B_j^1) \tag{1}$$

for $\tilde{v}_j(\mathbf{x}) = r \sum_j \widetilde{Q}_{j,k} v_k(\mathbf{x})$. Consider the RHS of (1). The integrand takes the value $x_j - B_j^1$ when $B_j^1 \leq b$ and the value $\tilde{v}_j(\mathbf{x})$ when $B_j^1 > b$ and so can be maximized by choosing b so that the event $B_j^1 \leq b$ occurs iff $x_j - B_j^1 \geq \tilde{v}_j(\mathbf{x})$ or equivalently $B_j^1 \leq x_j - \tilde{v}_j(\mathbf{x})$. This can be uniquely achieved by setting $b = x_j - \tilde{v}_j(\mathbf{x})$, since then $B_j^1 \leq b \Rightarrow B_j^1 \leq x_j - \tilde{v}_j(\mathbf{x})$ and $B_j^1 > b \Rightarrow B_j^1 > x_j - \tilde{v}_j(\mathbf{x})$. Moreover when $x_j - \tilde{v}_j(\mathbf{x})$ is interior to the support of G_j^1, the best response is itself unique, since with positive probability B_j^1 will occur in a neighborhood of $x_j - \tilde{v}_j(\mathbf{x})$, making it necessary (rather than just sufficient) to bid $x_j - \tilde{v}_j(\mathbf{x})$.

To argue continuity of the bidding strategies, consider two types $\mathbf{x}^1$ and $\mathbf{x}^2$ in some state j, and suppose wlog $v_j(\mathbf{x}^2) > v_j(\mathbf{x}^1)$. Suppose the optimal strategy for type $\mathbf{x}^2$ is $\beta(\mathbf{x}_2) = \mathbf{b}_2$ and suppose that type $\mathbf{x}^1$ sub-optimally bids according to $\mathbf{b}_2$. $\mathbf{x}^1$ will obtain the same allocations and make the same payments in expectation as $\mathbf{x}^2$, since their behavior is identical. Let the expected utility from this strategy be U^1 for type $\mathbf{x}^1$ and $U^2 = v_j(\mathbf{x}^2)$ for type $\mathbf{x}_2$ where the equality holds since the second type is playing optimally. Now $U^2 - U^1 \leq \max_j |x_j^2 - x_j^1|$ where the inequality holds since the difference in expected utility is upper bounded by the maximum difference in expected utility conditional on exiting by winning a particular product (or losing); and the RHS is exactly that maximum (if exit is by winning product j, expected payments are the same, and the difference in payoffs is exactly $x_j^2 - x_j^1$). But $0 < v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1) < v_j(\mathbf{x}^2) - U^1 = U^2 - U^1 \leq \max_j |x_j^2 - x_j^1|$. where the second inequality uses the fact that $U^1 < v_j(\mathbf{x}^1)$, since $v_j(\mathbf{x}^1)$ is obtained by playing optimally. By combining similar arguments one can show that $\max_j |v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1)| \leq \max_j |x_j^2 - x_j^1|$. This immediately proves that the continuation values are continuous in $\mathbf{x}$, with a modulus of continuity of 1: if $\max_j |x_j^2 - x_j^1| \leq \varepsilon$ then $\max_j |v_j(\mathbf{x}^2) - v_j(\mathbf{x}^1)| \leq \varepsilon$. Since bids are equal to valuations less continuation values, we can write: $\max_j |b_j^2 - b_j^1| = \max_j |x_j^1 - \tilde{v}_j(\mathbf{x}^1) - x_j^2 + \tilde{v}_j(\mathbf{x}^2)| \leq \max_j |x_j^1 - x_j^2| + \max_j |\tilde{v}_j(\mathbf{x}^1) - \tilde{v}_j(\mathbf{x}^2)| \leq 2 \max_j |x_j^1 - x_j^2$ where the first inequality follows by the triangle inequality. Thus bids are continuous in $\mathbf{x}$ with a modulus of continuity of 2.

For monotonicity, consider two types $\mathbf{x}^1$ and $\mathbf{x}^2$ with $x_k^1 = x_k^2$ for all $k \neq j$ and $x_j^1 < x_j^2$. Now $v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1) \leq x_j^2 - x_j^1$ for all k, since $\mathbf{x}^1$ can follow the strategy $\beta(\mathbf{x}^2)$ and get the exact same payoff as type $\mathbf{x}^2$ except when bidding on product j, when they get a payoff at most $x_j^2 - x_j^1$ lower. Thus: $\beta_j(\mathbf{x}^2) - \beta_j(\mathbf{x}^1) = x_j^2 - x_j^1 - r \sum_k \widetilde{Q}_{j,k}(v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1)) \geq x_j^2 - x_j^1 - r(x_j^2 - x_j^1) > 0$ proving that β_j is strictly increasing in x_j. Also $v_k(\mathbf{x}^2) - v_k(\mathbf{x}^1) \geq 0$ for all k since $\mathbf{x}^2$ can follow the the optimal strategy $\beta(\mathbf{x}^1)$ and get at least as high a payoff. It follows that $\beta_k(\mathbf{x}^2) - \beta_k(\mathbf{x}^1) = -r \sum_l \widetilde{Q}_{k,l}(v_l(\mathbf{x}^2) - v_l(\mathbf{x}^1)) \leq 0$ which (together with analysis of other cases) shows that β_j is decreasing in x_k for any $k \neq j$. □

Lemma 6. *For any strategy β, there exists a unique corresponding ergodic measure μ^β, converged to at geometric rate. This measure is absolutely continuous with respect to Lebesgue measure except at the atoms $\omega_j = (j, -1, -1, \ldots - 1), j \in -1 \cup 1 \ldots J$.*

Proof. Let ω index states, Ω be the state-space, $\mathcal{B}(\Omega)$ be the Borel σ-algebra over the state space, and $P_\beta(\omega, A)$ be the one-step transition probability when the players play according to β (i.e. the probability of reaching set $A \in \mathcal{B}(\Omega)$ from state ω in a single period). By Theorem 11.12 in Stokey et al. (1989), uniform geometric convergence in total variation norm will be achieved if their "condition M" holds: $\exists \varepsilon > 0$ such that for every $A \in \mathcal{B}(\Omega)$, either $[P_\beta(\omega, A) > \varepsilon \; \forall \omega \in \Omega]$ or $[P_\beta(\omega, A^c) > \varepsilon \; \forall \omega \in \Omega]$. We claim the following is sufficient for condition M: there exists some $\omega_0 \in S$ and some $\varepsilon > 0$ such that $P_\beta(\omega, \omega_0) > \varepsilon \; \forall \omega \in \Omega$. To prove this, notice that for any $A \in \mathcal{B}(\Omega)$ either $\omega_0 \in A$ or $\omega_0 \in A^c$. If the former, then for any $\omega \in \Omega$, $P_\beta(\omega, A) \geq P_\beta(\omega, \omega_0) > \varepsilon$. If the latter, then $P_\beta(\omega, A^c) \geq P_\beta(\omega, \omega_0) > \varepsilon$.

Let ω_0 be the state where there are no bidders and no supply (i.e. $(-1, -1 \cdots -1)$). This state is reachable in a single step: if at the end of a period everyone exits and no-one enters, and in the following period no products is available for auction, the state occurs. The probability of this occurring is at least $\underbrace{(1-r)^{N_{t-1}-1}}_{\text{all losers exit}} \underbrace{e^{-\lambda}}_{\text{no entry}} \underbrace{\min_j Q_{j,J+1}}_{\text{no supply}}$. So the required condition holds with $\varepsilon = (1-r)^{N_{t-1}-1} e^{-\lambda} \min_j Q_{j,J+1}$.

Next, by the Lebesgue decomposition theorem, it will suffice to show that the singular part of the ergodic measure μ^β consists only of atoms at the points ω_j. We assume that the singular part is discrete, and argue by contradiction: suppose there are other atoms and let the state $\tilde{\omega}$ be a selection from the set of atoms with the property that there are no other atomic states with more incumbent bidders. By the definition of ergodicity $\mu_j^\beta(\tilde{\omega}) = T(\mu_j^\beta, \beta)(\tilde{\omega})$ and because $\tilde{\omega}$ is an atom $\mu_j^\beta(\tilde{\omega}) > 0$, so $T(\mu_j^\beta, \beta)(\tilde{\omega}) > 0$. But for $T(\mu_j^\beta, \beta)(\tilde{\omega}) > 0$ it is necessary that there are other atoms that reach $\tilde{\omega}$ in a single step with strictly positive probability. Notice that no state with the same or fewer incumbent bidders than $\tilde{\omega}$ reaches $\tilde{\omega}$ with positive probability, since at least one incumbent bidder must exit (the winner), and then reaching $\tilde{\omega}$ requires drawing entrants with the exact same valuations as the exiting inbumbents. Since $\mathbf{F}$ is atomless wrt Lebesgue measure, this is a zero probability event. By similar logic, no state ω' with more bidders reaches $\tilde{\omega}$ with positive probability *unless* it includes all the incumbents in $\tilde{\omega}$ plus some additional incumbents (in this case there is a positive probability that these additional incumbents exit and no one enters, reaching $\tilde{\omega}$). But since by assumption $\tilde{\omega}$ was the atom with the largest number of incumbent bidders, any such state ω' must have zero measure under μ^β. But then we have shown that $T(\mu_j^\beta, \beta)(\tilde{\omega}) = 0$, a contradiction. □

Proof of Theorem 1.

Proof. Lemma 6 directly above establishes the existence of a unique ergodic measure for *any* strategy β. Thus to prove existence it will suffice to show that there is *some* strategy β^e that satisfies the optimality and consistency conditions laid out in Definition 1. Let $C(\mathcal{X})$ be the set of all continuous functions on $\mathcal{X}$, metrized by the sup norm. Let $\mathcal{L}_M = \{f \in C(\mathcal{X}) : L(f) \leq M\}$ where $L(f)$ is the Lipschitz constant of the function f, and $M \geq 2$ is finite. Define the best response function $\Gamma(\beta)$ to any strategy $\beta \in \mathcal{L}_M$ as in Lemma 1, i.e. $\Gamma(\beta)(x) = x_j - r\sum_k v_k(\mathbf{x})$. A fixed point $\beta = \Gamma(\beta)$ is a pure strategy equilibrium. We will prove the existence of a fixed point by Schauder's fixed point theorem: if Γ is a continuous mapping from a non-empty convex and compact subset K of a Banach space X into itself, then it has a fixed point.

Convexity, Compactness, Containment. $\mathcal{L}_M$ is non-empty and convex. Moreover, since $\mathcal{L}_M$ has a finite Lipschitz constant, the set of functions $f \in \mathcal{L}_M$ are uniformly equicontinuous, and so by Arzelà-Ascoli, $\mathcal{L}_M$ is relatively compact. Since $\mathcal{L}_M$ is also complete it is compact. It remains to prove that $\Gamma(\mathcal{L}_M) \subseteq \mathcal{L}_M$. Now, every $\beta \in \Gamma(C(\mathcal{X}))$ admits a modulus of continuity of 2, by the argument made in the proof of Lemma 1. This implies $\Gamma(\mathcal{L}_M) \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_M$ for $M \geq 2$.

Continuity. Next, we need to show that Γ is continuous in β when $\beta \in \mathcal{L}_M$ i.e. if $\beta^n \to \beta$ then $\Gamma(\beta^n) \to \Gamma(\beta)$. The first step is to show that the distribution of highest bids G_j^1 is continuous in β in the weak-* topology i.e. $\beta_n \to \beta$ implies $\mu_j^n \to \mu_j$ in the sense of weak convergence of measures (convergence in the weak topology is equivalent in this case). By Lemma 7 directly below, the ergodic distribution of types is weakly continuous in β. Now given an ergodic measure μ^β the highest bid distribution G_j^1 is defined by:

$$G_j^1(b) = \frac{\mathbb{P}_{\mu^\beta}\left(\max_{i>1} \beta_j(\omega_i) \leq b \wedge \omega_1 = j\right)}{\mathbb{P}_{\mu^\beta}\left(\omega_1 = j\right)}$$

since the first entry ω_1 indicates the product under auction and by definition the highest bid is $\max_{i>1} \beta_j(\omega_i)$ (where we adopt the convention $\beta_j(-1) = -\infty \; \forall j$ i.e. non-existent bidders don't count towards the maximum). This is continuous in β since max is a continuous operation, and by Lemma 6, μ^β is absolutely continuous wrt Lebesgue measure in any states where there are incumbent bidders in the market.

The second step is to show that if the highest bid distributions $\mu = \{\mu_j\}$ are close in the sense of weak convergence, then so are the best responses. Recall from the main text that when playing optimally (i.e. best responding) the value of the game is just the value of an infinite sequence of annuities, each of which pays $\mathbb{E}_{G_j^1}[\max\{0, \beta_j(\mathbf{x}) - B_j^1\}]$ whenever j is auctioned. The term inside the expectation is a continuous function, and so it follows from the definition of weak convergence that the value function $v_j(\mathbf{x})$ is smooth in μ. Then since the best responses $\Gamma(\beta)(x)$ are by definition equal to value less discounted continuation value, they too must be smooth in the bid distributions. Putting this all together, we get the required continuity of Γ in β.

Uniqueness. We prove uniqueness in the case $J = 1$ by constructing a contraction mapping. Let β and β' be two equilibrium bidding strategies, necessarily monotone and continuous by earlier results. By definition of equilibrium, $\Gamma(\beta) = \beta$ and $\Gamma(\beta') = \beta'$. Let $v(x, \beta)$ and $v(x, \beta')$ be the corresponding value functions (dropping the vector notation since there is a single product, and explicitly showing dependence on the strategies β). Γ is a contraction:

$$\begin{aligned}
\|\Gamma(\beta) - \Gamma(\beta')\| &= \max_x (x - rv(x,\beta)) - (x - rv(x,\beta')) \\
&< \max_x |G^1_\beta(\beta(x))\left(x - \mathbb{E}_\beta[B^1|B^1 \leq \beta(x)]\right) \\
&\quad - G^1_{\beta'}\left(\beta'_j(x)\right)\left(x - \mathbb{E}_{\beta'}[B^1|B^1 \leq \beta'(x)]\right)| \\
&\leq \max_x G^1_\beta(\beta(x))\left|\mathbb{E}_\beta[B^1|B_1 \leq \beta(\mathbf{x})] - \mathbb{E}_{\beta'}[B_1|B_1 \leq \beta'(x)]\right| \\
&\leq \|\beta - \beta'\|
\end{aligned}$$

where in the third line we use the fact that $G^1_\beta(\beta(x)) = G^1_{\beta'}(\beta'(x))$ for all x. This property holds because under both strategies the same winner is chosen in each auction (types are totally ordered), so that the distribution of highest bids is equal across strategies if evaluated at the bids of a fixed type x in any state s. Therefore Γ is a contraction which guarantees uniqueness of any fixed point (and a fixed point exists via the above application of Schauder's fixed point theorem.). □

Lemma 7. *Let $\beta \in \mathcal{L}_M$. The ergodic measure μ^β is weakly continuous in β i.e. for all sequences $(\beta_n) \in \mathcal{L}_M$ converging to $\beta^* \in \mathcal{L}_M$ we have : $\mu^{\beta_n}(\cdot) \xrightarrow{\text{weakly}} \mu_{\beta^*}(\cdot)$.*

Proof. Let (β_n) be a sequence in $\mathcal{L}_M$ that converges to $\beta^* \in \mathcal{L}_M$. Towards a contradiction suppose that (μ^{β_n}) does not converge to μ^{β^*}. Now since the state space Ω is compact, the set of measures $(\mu_\beta)_{\beta\in\Omega}$ are tight. Moreover $\mathcal{L}_M$ is compact, and so by Prokhorov's theorem the set of measures $(\mu^\beta)_{\beta\in\mathcal{L}_M}$ is sequentially compact. Therefore we know that there is a subsequence (β_{m_n}) and an element $\beta' \in \mathcal{L}_M$ such that

$$\mu_{\beta_{m_n}} \xrightarrow{weakly} \mu_{\beta'}.$$

Let $d = \bar{N} + 1$ and let $A \in \mathbb{R}^d$ be a Borel set. We define $P_\beta(\omega, A)$ as the probability of reaching the set A from the state ω in a single step when strategies β are played. Let $\epsilon > 0$

be a positive real. We have

$$\begin{aligned}
&\Big|\mu_{\beta_{m_n}}(A) - \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta_{m_n}}(\omega)\Big| \\
&\overset{(a)}{\leq} \Big| \int_{\mathbb{R}^d} \Big[P_{\beta_{m_n}}(\omega, A) - P_{\beta^*}(\omega, A)\Big] d\mu_{\beta_{m_n}}(\omega)\Big| \\
&\overset{(b)}{\leq} \epsilon + \mathbb{P}_{\omega\sim\mu_{\beta_{m_n}}}(\big|[P_{\beta_n}(\omega, A) - P_{\beta^*}(\omega, A)\big| > \epsilon).
\end{aligned}$$

where to get (a) we used the fact that $\mu_{\beta_{m_n}}$ is the ergodic measure of $P_{\beta_{m_n}}$, and (b) splits the realization of the state into small and large deviations.

Next, we will show that $\mathbb{P}_{\omega\sim\mu_{\beta_{m_n}}}(\big|[P_{\beta_{m_n}}(\omega, A) - P_{\beta^*}(\omega, A)\big| > \epsilon) \to 0$ as $\beta_{m_n} \to \beta^*$ (since $\beta_n \to \beta^*$, the subsequence does too). It is sufficient to show that $P_\beta(\omega, A)$ is continuous in β almost everywhere. Notice that next period's state depends only on the strategies β through determining the winner of the current auction — entry, activity and exit through losing are constant in β. Fix a state ω. Then the identity of the winner is continuous in β at ω, unless there is a tie since then a small pertubation in β can affect the winner (and therefore the next state) discontinuously. But the set of such states is of lower dimension than the state space, and therefore Lebesgue measure zero, and (by absolute continuity of the ergodic measures) zero probability. It follows that:

$$\Big|\mu_{\beta_{m_n}}(A) - \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta_{m_n}}(\omega)\Big| \to 0. \tag{2}$$

Again using the fact that all the ergodic measures are assumed to be absolutely continuous with respect to the Lebesgue measure we have that: $\mu_{\beta_{m_n}}(A) \to \mu_{\beta'}(A)$. Moreover as $\omega \to P_{\beta^*}(\omega, A)$ is measurable and bounded we know that for all $\delta > 0$ there are N_δ different Borel sets (B_i) and different reals c_i such that

$$\sup_\omega \big|P_{\beta^*}(\omega, A) - \sum_{i\leq N_\delta} c_i \mathbb{I}(\omega \in B_i)\big| \leq \delta$$

Moreover using the fact that $\mu_{\beta_{m_n}}(\cdot)$ converges weakly to $\mu_{\beta'}(\cdot)$ we have:

$$\int_{\mathbb{R}^d} \big[\sum_{i\leq N_\delta} c_i \mathbb{I}(\omega \in B_i)\big] d\mu_{\beta_{m_n}}(x) - \int_{\mathbb{R}^d} \big[\sum_{i\leq N_\delta} c_i \mathbb{I}(\omega \in B_i)\big] d\mu_{\beta'}(x) \to 0.$$

Therefore by a triangle inequality we have:

$$\int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta_{m_n}}(\omega) - \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta'}(\omega) \to 0. \tag{3}$$

Now since $\mu_{\beta_{m_n}}(A)$ weakly converges to $\mu_{\beta'}(A)$, we can combine this fact with (2) and (3) to deduce that for all Borel sets A we have

$$\mu_{\beta'}(A) = \int_{\mathbb{R}^d} P_{\beta^*}(\omega, A) d\mu_{\beta'}(\omega) \tag{4}$$

But by Lemma 6 the ergodic measure is unique, so we must have that: $\mu_{\beta'} = \mu_{\beta^*}$. Contradiction. □

Proof of Lemma 2.

Proof. Follows from the proof of Lemma 1, replacing j subscripts with functions of s. □

Proof of Lemma 3.

Proof. See the argument in the text. □

Proof of Lemma 4.

Proof. We begin by writing out an expression for the continuation value, prior to the realizations of the transient shocks.

$$
\begin{aligned}
v_j(\check{\mathbf{x}}) &= \int\int G_j^1(\check{b}_j + \xi + \varepsilon|\xi)\left(\check{x}_j + \xi + \varepsilon - E_{G_j^1|\xi}[B_j^1|B_j^1 < \check{b}_j + \xi + \varepsilon, \xi]\right) dF_\varepsilon(\varepsilon) dF_\xi(\xi) \\
&+ \int\int (1 - G_j^1(\check{b}_j + \xi + \varepsilon|\xi)) dF_\varepsilon(\varepsilon) dF_\xi(\xi) \left(r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}})\right) \\
&= \int\int \check{G}_j^1(\check{b}_j + \varepsilon)\left(\check{x}_j + \varepsilon - E_{\check{G}_j^1}[\check{B}_j^1|\check{B}_j^1 < \check{b}_j + \varepsilon]\right) dF_\varepsilon(\varepsilon) dF_\xi(\xi) \\
&+ \int\int (1 - \check{G}_j^1(\check{b}_j + \varepsilon)) dF_\varepsilon(\varepsilon) dF_\xi(\xi) \left(r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}})\right) \\
&= \int \check{G}_j^1(\check{b}_j + \varepsilon)\left(\check{x}_j + \varepsilon - E_{\check{G}_j^1}[B_j^1|B_j^1 < \check{b}_j + \varepsilon]\right) dF_\varepsilon(\varepsilon) \\
&+ \int (1 - \check{G}_j^1(\check{b}_j + \varepsilon)) dF_\varepsilon(\varepsilon) \left(r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}})\right)
\end{aligned} \tag{5}
$$

The first line is the definition of the continuation value, as the probability of winning times the surplus conditional on winning, plus the discounted continuation value when losing, integrated out over the realizations of both shocks. In this expression the distribution of competing bids conditions on the commonly known shock ξ_l, which affects the bids of all bidders. The second line follows on noting that since ξ_l is common knowledge, all bidders bidding on item l will increase their bids by exactly ξ_l relative to an auction where the item shock is zero, and so we can work with the checked distributions instead. The final line removes the redundant outer integral.

Now — in this case by definition — the bid types are equal to the valuation less the discounted continuation value: $\check{b}_j = \check{x}_j - r \sum_k \widetilde{Q}_{j,k} v_k(\check{\mathbf{x}})$. Solving for $\check{x}_j$ in the above expression and substituting gives:

$$\begin{aligned}
v_j(\check{\mathbf{x}}) &= \int \check{G}_j^1(\check{b}_j+\varepsilon)\left(\check{b}_j + r\sum_k \widetilde{Q}_{j,k}v_k(\check{\mathbf{x}}) + \varepsilon - E_{\check{G}_j^1}[B_j^1|B_j^1 < \check{b}_j+\varepsilon]\right) dF_\varepsilon(\varepsilon) \\
&+ \int (1-\check{G}_j^1(\check{b}_j+\varepsilon))dF_\varepsilon(\varepsilon)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\check{\mathbf{x}})\right) \\
&= \int \check{G}_j^1(\check{b}_j+\varepsilon)\left(\check{b}_j + \varepsilon - E_{\check{G}_j^1}[B_j^1|B_j^1 < \check{b}_j+\varepsilon]\right) dF_\varepsilon(\varepsilon) \\
&+ \int (\check{G}_j^1(\check{b}_j+\varepsilon) + 1 - \check{G}_j^1(\check{b}_j+\varepsilon))dF_\varepsilon(\varepsilon)\left(r\sum_k \widetilde{Q}_{j,k}v_k(\check{\mathbf{x}})\right) \\
&= u_j(\check{\mathbf{b}}) + r\sum_k \widetilde{Q}_{j,k}v_k(\check{\mathbf{x}})
\end{aligned} \tag{6}$$

where in the second line we collect terms in $r\sum_k \widetilde{Q}_{j,k}v_k(\check{\mathbf{x}})$, and in the final line we use the fact that $u_j(\cdot)$ is a function of the bid-type only. For each state j (including the no-supply state) we have an equation of this form. Stacking the equations we get a system of equations:

$$v(\check{\mathbf{x}}) = u(\check{\mathbf{b}}) + r\widetilde{Q}v(\check{\mathbf{x}})$$

where $v(\check{\mathbf{x}})$ is $J \times 1$, $u(\check{\mathbf{b}})$ is $J \times 1$, r is a scalar and $\widetilde{Q}$ is $J \times J$. Solving the system yields:

$$v(\check{\mathbf{x}}) = (I - r\widetilde{Q})^{-1}u(\check{\mathbf{b}})$$

where since $r \in (0,1)$ and $\widetilde{Q}$ is right-stochastic, the matrix $(I - r\widetilde{Q})$ is strictly dominant diagonal and therefore invertible. Re-arranging the definition of the bid types, the underlying valuation is given by $\check{\mathbf{x}} = \check{\mathbf{b}} + r\widetilde{Q}v(\check{\mathbf{x}})$. Substituting in for the continuation value:

$$\check{\mathbf{x}} \equiv \zeta(\check{\mathbf{b}}) = \check{\mathbf{b}} + r\widetilde{Q}(I - r\widetilde{Q})^{-1}u(\check{\mathbf{b}})$$

completing the proof. □

Proof of Lemma 5.

Proof. We will repeatedly invoke (a minor modification of) Lemma 2 of Evdokimov and White (2012): suppose the joint distribution of (Y_1, Y_2) is observed, $Y_1 = M + U_1$ and

$Y_2 = M + U_2$ and (M, U_1, U_2) are mutually independent, with $E[|Y_1| + |Y_2|] < \infty$ and either $E[M] = 0$ or $E[U_1] = 0$. Suppose also that there exist positive constants c_1 and c_2 such that $f_{U_1}(u) < c_1 \exp(-c_2\|u\|)$. Then the distributions of M, U_1 and U_2 are identified.

Take as Y_1 and Y_2 the joint distribution of pairs of bids placed by bidders that have just entered the market. From (10), these bids are a sum of two components: a component that is common (ξ_l) and a component that is bidder-specific ($\check{x}_{i,j} - r\sum k\widetilde{Q}_{j,k}v_k(\check{\mathbf{x}}_i) + \varepsilon_{i,t}$). Because ξ_l, $\varepsilon_{i,l}$ and $\mathbf{x}_i$ are mutually independent, with finite second moments, the two components are mutually independent with finite absolute first moments (the latter following from Cauchy-Schwarz). Take $U_1 = \check{x}_{i,j} - r\sum k\widetilde{Q}_{j,k}v_k(\check{\mathbf{x}}_i) + \varepsilon_{i,t}$. It satisfies the thin tails condition, since the valuations have finite support and by assumption $\varepsilon_{i,t}$ satisfies the thin tails condition. Finally, taking $M = \xi_l$ we have $E[M] = 0$. So the lemma's conditions are satisfied and we can identify F_ξ.

Next, consider the joint distribution of pairs of bids of the same bidder in two successive auctions of the same item. Again from (10), these are composed of a persistent bidder-specific element, $M = \check{x}_{i,j} - r\sum k\widetilde{Q}_{j,k}v_k(\check{\mathbf{x}}_i) + \xi_l$, and idiosyncratic shocks $U_1 = \varepsilon_{i,t_1}$ and $U_2 = \varepsilon_{i,t_2}$. Again all the conditions of the theorem are satisfied, with $E[U_1] = 0$ this time, allowing identification of F_ε.

Finally, notice that $B^1_{j,l,t} = \check{B}^1_{j,t} + \xi_l$, where now the distribution of the LHS G^1_j and the distribution of F_ξ are known, and $\check{B}^1_{j,t}$ and ξ_l are independent. Then since by assumption the characteristic function of the distribution F_ξ has isolated real zeros $\check{B}^1_{j,t}$ is thus identified as per the remark in footnote 22. Finally, given that all these objects are identified, $u_j(\check{\mathbf{b}})$ is identified according to (11). □

Lemma 8. *The function* $s(\check{\mathbf{b}})$ *is identified.*

Proof. We provide a procedure for iteratively computing the selection correction function, providing a constructive proof of identification. Bidding on each of the products $j = 1 \dots J$ is a transitory state, as is bidding in a period with no supply. Exit is an absorbing state. The probability of transition from any transitory state to another one is $r_j(\check{\mathbf{b}})\widetilde{Q}_{j,k}$; the probability of exit is $(1 - r_j(\check{\mathbf{b}}))$. All of these are identified from knowledge of $\widetilde{Q}$, F_ε and $\{\check{G}^1_j\}$. Define for some subset A of the transient states $1 \dots J+1$, $p(\check{\mathbf{b}}, A, j)$ to be the probability that bid type $\check{\mathbf{b}}$ only visits states *within* A prior to exit, when entering in state j. We have the following recursive representation:

$$p(\check{\mathbf{b}}, A, j) = 1(j \in A) \left((1 - r_j(\check{\mathbf{b}})) + r_j(\check{\mathbf{b}}) \sum_k \widetilde{Q}_{j,k} p(\check{\mathbf{b}}, A, k) \right)$$

since the probability of staying in A requires (i) starting in A (hence the indicator) and (ii) either exiting or surviving and entering another state k and once again having the chance of staying within A given by $p(\check{\mathbf{b}}, A, k)$. This recursive expression satisfies the Blackwell conditions for a contraction, and so can be identified and quickly computed for any A.

Now the probability of visiting *every* state in a set A, denoted $P(\check{\mathbf{b}}, A, j)$, is equal to the probability of staying within A less the probability of staying within any strict subset of A:

$$P(\check{\mathbf{b}}, A, j) = p(\check{\mathbf{b}}, A, j) - \sum_{B \subset A} p(\check{\mathbf{b}}, B, j)$$

Given this, we can compute $P(\check{\mathbf{b}}, \mathcal{J}, j)$ for each j, and then sum over the steady-state probability of entering in each of the states j to get $s(\check{\mathbf{b}}) = \sum_{j=1}^{J} \pi_j P(\check{\mathbf{b}}, \mathcal{J}, j)$. □

Proof of Theorem 2.

Proof. We supplement the argument from the main text with additional details. First, we argue that the ergodic distribution $\mathbf{G}(\mathbf{b})$ is identified. We can rewrite it as follows:

$$\begin{aligned} \mathbf{G}(\mathbf{b}) = \mathbb{P}(\mathbf{B} \leq \mathbf{b} | \mathbf{B} \text{ is complete}) =& \frac{\mathbb{P}(\mathbf{B} \leq \mathbf{b} \ \wedge \mathbf{B} \ \text{ is complete})}{\mathbb{P}(\mathbf{B} \text{ is complete})} \\ &= \lim_{T \to \infty} \frac{\frac{1}{T} \sum_t 1(\mathbf{B}_t \leq \mathbf{b} \ \wedge \mathbf{B}_t \ \text{ is complete})}{\frac{1}{T} \sum_t 1(\mathbf{B}_t \ \text{ is complete})} \end{aligned}$$

where $\mathbf{B}$ is a random bid vector and $\mathbf{B}_t$ is sampled by selecting a bidder at random from each cohort of entrants, and then tracking their bids until they exit the market. The first two equalities in the display follow by definition and the definition of conditional probabilities. The last equality follows from the pointwise ergodic theorem: the averages in the numerator and the denominator tend asymptotically to the corresponding probabilities with respect to the ergodic measure of market states ω defined in Lemma 6. By expanding it in this way we can see the empirical objects required to estimate $\mathbf{G}(\mathbf{b})$: the probability that a randomly selected bidder from a cohort has a complete bid vector less than $\mathbf{b}$ and the probability that a randomly selected bidder has a complete bid vector, both of which are observable.

Second, we work through the deconvolution argument. $\mathbf{G}$ is a convolution of $\widetilde{\mathbf{F}}^S$ and $\mathbf{F}_\xi$ and $\mathbf{F}_\varepsilon$, where the (selected) bid-type $\check{\mathbf{b}}$ and $\boldsymbol{\xi}$ and $\boldsymbol{\varepsilon}$ are all mutually independent. Letting $\phi_X(\cdot)$ denoted the characteristic function of the random variable X, it follows that we have:

$$\phi_{\mathbf{b}}(t) = \phi_{\check{\mathbf{b}}}(t)\phi_{\boldsymbol{\xi}}(t)\phi_{\boldsymbol{\varepsilon}}(t)$$

Re-arranging yields:

$$\phi_{\check{\mathbf{b}}}(t) = \frac{\phi_{\mathbf{b}}(t)}{\phi_{\boldsymbol{\xi}}(t)\phi_{\boldsymbol{\varepsilon}}(t)}$$

where the numerator and denominator are both known. Moreover by the assumptions of Theorem 2 the denominator has only isolated real zeros so that the characteristic function $\phi_{\check{\mathbf{b}}}(t)$ is integrable with respect to Lebesgue measure. Integrating the characteristic function of $\check{\mathbf{b}}$ permits recovery of the distribution $\widetilde{\mathbf{F}}^S$. Finally, the density of bid-types is $\tilde{f}(\check{\mathbf{b}}) = k\tilde{f}^S(\check{\mathbf{b}})/s(\check{\mathbf{b}})$ where k is the observable probability that a randomly chosen bidder submits a complete bid. Finally, by Lemma 4 each bid-type can be inverted to recover the underlying valuation, and we have:

$$\mathbf{F}_X(x) = \mathbb{P}_{\widetilde{\mathbf{F}}}(\{\check{\mathbf{b}} : \zeta(\check{\mathbf{b}}) \leq \mathbf{x}\})$$

□

Proof of Corollary 1.

Proof. From (15) the permanent part of the each bidder's valuation is $\mathbf{x} = \boldsymbol{\alpha}W$ where α is sampled from $\mathbf{F}_\alpha$. Now since W has full rank, we can write $\boldsymbol{\alpha} = W^+\mathbf{x}$ where W^+ is the Moore-Penrose pseudo-inverse of W. So the mapping is invertible: i.e. each α maps to a unique x. It follows that the densities of $\mathbf{F}_\alpha$ and $\mathbf{F}$ are related: $f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = f_{\mathbf{x}}(W\alpha)$. Now from Theorem 2 the distribution $\mathbf{F}$ is identified, so we are done. □

B Empirical Appendix (not for publication)

B.1 Data Source

We used data from eBay auctions for compact cameras that was purchased from Terapeak, a private company that uses eBay data to offer analytics tools for sellers on the platform. Our data included item, seller, and bidder attributes as documented in Table B-1 from the main text.

B.2 Sample Construction

We restricted attention to auctions for *new* compact cameras that ended between the dates of Feb 5 and May 6, 2007. By way of data cleaning we also imposed the following restrictions:

1. We were somewhat concerned that we may have been observing some "shill" bidding by sellers intent on raising revenues. We used the following rather coarse procedure to detect such bidding behavior: for any given bidder, if they won at least five auctions and at 80% of them were from the same seller, then we flagged them as a shill bidder. We excluded all auctions in which a so-designated shill bidder was among the set of observed bidders.

2. There are many senses in which a bid may be an "outlier". We excluded all winning bids that were more than twice the sale price of the listing (either the second-highest bid or the reserve price, whichever is greater).

3. Though we did not exclude auctions that used the Auction-Buy-it-Now feature, we did exclude auctions that ended with the execution of the BIN option. This option disappears once the first bid is made, so those listings function as regular auctions afterwards. See Ackerberg et al. (2009) for a fuller treatment of Auction-Buy-it-Now auctions.

4. We drop listings for which we are missing data on attributes (e.g., zoom, resolution), for which those attributes are unreasonable or technically impossible, or where the product line is not indicated.

B.3 Descriptive evidence

Our sample construction leaves us with a dataset of 19160 auctions, 4387 sellers and 74375 unique bidders. Summary statistics are presented in Table B-1. In the top panel, we summarize the data at an auction level (i.e. an observation is an auction). The average gap between the winning bid and the closing price is $10.60. In a static auctions framework auctions this would be a direct estimate of the average consumer surplus of winning bidders, but we will show that it is a substantial underestimate once continuation value is taken into account. The probability of sale is quite high, at 0.95. Those products that do not sell tend to be among the small handful which employ “secret reserve prices”, whereby the item only sells if the highest bid meets a reserve price set by the seller but hidden from bidders. On average an auction attracts 7 – 8 unique bidders and 16 – 17 bids, though there is substantial variance.

Table B-1 also summarizes the data at the seller and bidder level. There are just over four listings for every seller in our marketplace. The distribution of seller shares is skewed, with large, experienced sellers making up the bulk of listings. However there are many such large sellers and the seller concentration measure (HHI) remains low at 0.04. On the bidder side we find some motivation for the assumptions of our model. Most bidders are unsuccessful in acquiring an item (the purchase rate is 24.3%), but they are active in the market for nearly a week and on average are observed bidding in two different auctions, with some participating in many more than that.[39] This repeated participation is not driven by multi-unit demand—99% of our bidders make one or fewer compact camera purchases.

Having introduced the data, we now offer some descriptive evidence that indicates our model is a reasonable approximation to the behavior we see in the market.

Do bidders substitute across products? If bidders generally bid on the same products repeatedly (or have identical preferences over product characteristics), one might think that the single-good models familiar from the existing empirical auctions literature may suffice. So to get some evidence on this, we look for evidence of substitution across products. There are many distinct products sold in our dataset, but in the demand system below we allow random coefficients over camera resolution, and so this is the relevant definition of a product for the purposes of our analysis. We therefore calculate a transition matrix across cameras

[39] We say a bidder is “active” from the start of the first auction they bid in to the end of the last auction.

Table B-1: Summary Statistics: Online Auctions of Compact Cameras

	Mean	Std. Dev.	Min	Max
Auction-Level Data				
Winning bid	228.2	99.58	24.51	2346.0
Closing price	219.3	90.72	20.26	2025
Shipping cost	18.45	9.116	0	150
Starting price	51.27	86.88	0.01000	564.0
Secret reserve?	0.0475	0.213	0	1
Item sold?	1	0	1	1
Bid count	16.71	9.931	1	95
Number of unique bidders	7.808	4.005	1	34
Camera resolution (megapixels)	6.710	1.206	5	10
Optical zoom	4.872	2.959	2	15
Digital zoom	4.246	1.225	2	10
Comes with accessories	0.527	0.499	0	1
Number of auctions	9213			
Seller-Level Data				
Number of listings	4.086	54.05	1	2188
Number of sales	3.839	51.57	0	2101
Seller feedback	801.6	7590.6	-1	251949.1
Number of sellers	2255			
Seller HHI	0.0804			
Bidder-Level Data				
Auctions participated in	1.848	2.860	1	136
Incumbent period (days)	2.423	5.056	-0.000744	69.68
Time between bids (days)	4.041	9.466	0	91.06
Number of purchases	0.222	0.416	0	1
$\leq$ 1 purchase	1	0	1	1
Bidder feedback	85.99	444.3	-999	60013
Winning bidder's age	36.43	16.25	1.742	95.98
Number of bidders	38919			
Buyer HHI	0.000116			

Notes: Summary statistics for the full dataset, which consists of all auctions of compact cameras auctions on a large online platform that ended during the period Feb 5 – May 6, 2007. Observations with missing product characteristics have been dropped. The "incumbent period" for a buyer is measured as the time from the start-date of the first auction bid in to the end of the last auction bid in. "Time between bids" is the gap between the first and second bids by a buyer on different objects, measured only for the subsample of bidders with multiple bids. "Seller HHI" is the Herfidahl–Hirschman index for sellers, based on their share of items sold. "Buyer HHI" is the analogous measure for buyers, based on the share of items bought.

of different resolution, shown in Table B-2. Each entry in the matrix is the probability (expressed as a percentage) that a bidder who bids on a camera of the resolution in a given row subsequently bids on a camera with the column resolution. We find that most buyers that bid on multiple cameras tend to bid on a product with the same resolution next, but the probability of this is far away from 100%, ranging from 64.8% to 80.8%. When they substitute, they tend to pick a product of similar resolution (the biggest off-diagonal elements are generally close to the diagonal).

Table B-2: Substitution Patterns for Repeat Bidders

	5MP	6MP	7MP	8MP	10MP
5MP	64.8	17.3	12.3	4.2	1.4
6MP	5.9	75.3	13.7	3.5	1.7
7MP	2.9	8.8	80.8	5.2	2.4
8MP	2.6	5.4	10.2	78.3	3.6
10MP	1.9	6.1	13.0	8.0	71.0

Notes: Each entry in the matrix gives the observed frequency with which a bidder who is observed bidding on the row product is next observed bidding on the column product.

Do sellers use reserve prices? In our empirical model, there are no reserve prices, although they are straightforward to implement in counterfactuals. In reality, there are two ways that sellers might implement a reserve price. The first is eBay's "secret reserve" feature. If a seller elects to use this option, the reserve price is shrouded and bidders are notified only of whether the standing high bid has exceeded the secret reserve or not. The pricing of this feature is relatively aggressive, which is reflected in the fact that it is rarely used. In our sample, less than five percent of sellers opt to employ a secret reserve price.

An alternative strategy to set a reserve price would be to set a high starting price. This does not incur additional fees. In practice, however, the vast majority of sellers do not do this. Figure B-2 depicts a kernel density estimate of the distribution of starting prices in our dataset; it is clear that, by and large, sellers do not employ anything approximating optimal reserves. This is demonstrably suboptimal, but perhaps not surprising: it is difficult to work out and motivate the use of optimal reserves even for professional sellers and advertising auction platforms, and so eBay sellers may prefer a simpler strategy. Moreover, the platform explicitly encourages to use low starting prices.

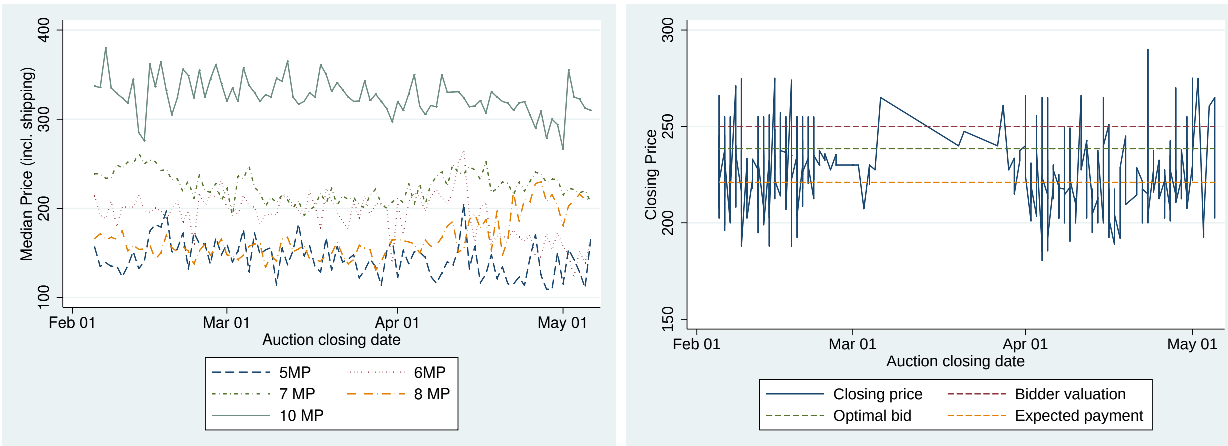

Figure B-1: Market Dynamics. The left panel shows the median daily price (including shipping) over the sample period, separately by camera resolution (4MP cameras omitted for clarity). The right panel shows the closing transaction prices on auctions of a Kodak Easyshare Z710 from a single seller over the sample period, superimposing over this the valuation, optimal bid and expected payment of a bidder with $250 valuation (see text for more details).

Do bidders have option value? Another focus of our approach has been to emphasize dynamic bidding and option values. Option values arise from price fluctuations—a bid that loses today may win tomorrow. Many economists have the intuition that in large markets the law of one price holds, and so price fluctuations should be minimal. This is not true here. The top left panel of Figure B-1 shows the median daily transaction prices (including shipping) on different models of cameras over the sample period. There is quite a lot of variability—a typical change is 10%—and since it is a median price, this is not driven entirely by outliers.

To rule out variation based on compositional effects, we drilled down to look at the highest volume seller's most popular product (a Kodak Easyshare Z710), and plotted the price series against time (fluctuations could be within a day), shown in the top right panel of Figure B-1. Prices vary from below $200 to over $250 dollars. Doing some rough back of the envelope calculations based on the observed price distribution, a bidder with a valuation of $250 (red dashed line), bidding once a day, with a daily exit hazard rate of 0.5, should optimally shade their bid down to $238 (green dashed line), because of the option value (and should expect to pay closer to $225, orange dashed line). Indeed this option value should be present in many online markets: Einav et al. (2015) have found the standard deviation in price to be on the order of 10% of the transaction price in most eBay categories.

We also observe that the average time between bids is just over four days, with many auctions

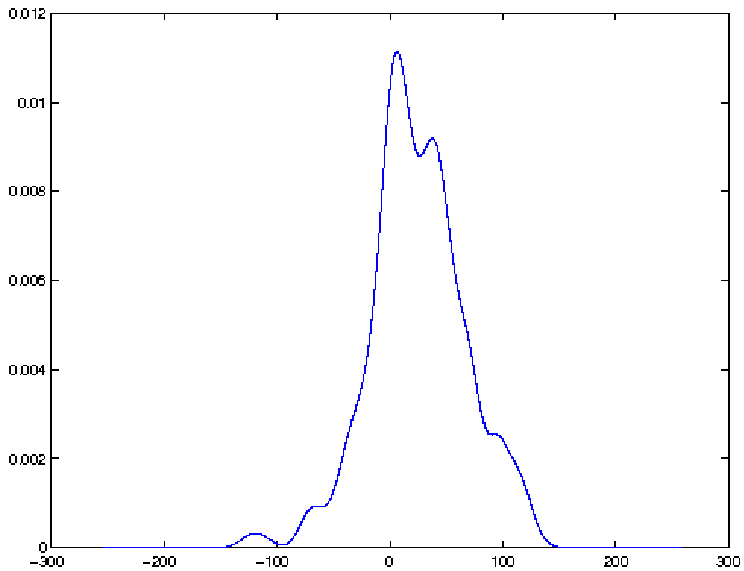

Figure B-2: Density of $\frac{b_{i,j_1}-b_{i,j_2}}{\text{res}_{i,j_1}-\text{res}_{i,j_2}}$. This figure presents a kernel density plot of the ratio between the difference in bids across cameras of two resolution levels and the difference in resolution levels, across bidders.

closing in the meantime. This suggests that bidders are inattentive (i.e. τ close to zero). In view of this, Assumption 1 seems quite reasonable: information from the current auction is probably a poor predictor of what conditions will be like the next time the bidder bids.

Is there unobserved product heterogeneity? The way we specify valuations below incorporates random coefficients over camera resolution, as well as common preferences for observable camera attributes (e.g. brand) and an unobservable component. To motivate the inclusion of this unobserved product heterogeneity, we perform a simple heuristic estimation exercise. We take bids that have been "normalized" to account for observable heterogeneity (we explain this process below), and look at bidders who have bid exactly twice on cameras at two different resolution levels (e.g. 7MP and 8MP cameras). Dividing the difference in their normalized bids by the difference in the camera resolutions gives us a crude estimate of this bidder's willingness to pay for megapixels, and the distribution of differences across such bidders gives us an idea of the diversity in bidders' willingness to pay.

Figure B-2 presents the density of that statistic computed at the bidder level for this restricted subset, without any kind of selection correction. While the mean of this distribution is positive (reassuringly) and the variance is large (motivating heterogeneous preferences for resolution), there is a troublesome and substantial mass to the left of zero. It is implausible that many consumers have negative marginal utility from higher resolution cameras, and we instead interpret this as evidence of unobserved product heterogeneity in our data that confounds this simple approach (we will offer more direct evidence below).

B.4 Alternative Empirical Designs

Our application was chosen to highlight the flexibility of the identification and estimation framework, but of course there are a number of alternative empirical designs that we could have employed consistent with that framework, designs which may be more or less appropriate given the aims of the estimation exercise. Here we discuss a few of them.

Aggregate Fluctuations. We found no evidence of aggregate fluctuations in supply or demand in our sample (see Figure B-1 and the corresponding discussion), but how costly would it be to allow for changes in market aggregates? Following the theory offered in Lemma 2, our model can accommodate these fluctuations by augmenting the state space of the model with "public states," which can evolve stochastically or deterministically. The cost of doing so turns out not to be imposed on the bidder-wise inversion or the dimension of the state-space—requirements for bidder-wise inversion depend only on the dimension of the type space. Instead, the practical constraint is that the econometrician must estimate $G^{(1)}$ and $G^{(2)}$ for every point in the augmented state space.

It is worth noting that the same procedure could be adopted to allow for other natural extensions in the eBay environment. For instance, variation in supply might translate to variation in how many close substitutes are available in the upcoming sequence of auctions, which would affect shading in a model with limited bidder foresight and patience (see our prior working paper, Backus and Lewis (2010), for a full treatment of this scenario).

Alternative Type Spaces. A costlier extension would be to allow bidders to have random preferences for each product type or brand. Analogous to the fixed-price demand literature, we have employed characteristic space (in this case, resolution) as dimension reduction.

The richness of dimension that one can accommodate in the type space should be dictated, following the intuition of our identification result, by the frequency with which one observes bidders in multiple states. We find a substantial population of bidders who bid on at least two different cameras with two different resolution values, and so we set the dimension to two and identify demand using that population. It is natural to ask, however, whether there is a way that we could have used the entire sample of bidders. Indeed we could have: since we have formally modeled sample selection, our model does produce moments that we can match about the distribution of partially observed bid vectors. We eschewed these in the exercise in order to stay closer to the nonparametric identification strategy. Alternatively, we could have specified a one-dimensional type space and performed the bidder-wise inversion for the entire sample. These are practical choices that will depend on the aim of the exercise—our aim is to illustrate and highlight the choices.

B.5 First-Stage Estimation

Estimation of supply and exit rates. Estimation begins with the recovery of the survival rate r, and the multinomial supply π. Note that with multinomial supply, the activity rate τ plays no role in optimal bidding and is therefore unnecessary for identification of demand. To see this, recall Lemma 1 and that $\widetilde{Q} \equiv \sum_{s=1}^{\infty} \tau(1-\tau)^{s-1} Q^s$. So, if the state tomorrow is independent of the state today, then $\tilde{Q} = Q$ and each row is simply π, which is also the stationary distribution. In contrast, with general Markov supply transitions Q, the activity rate drives a wedge between supply transitions and the state transitions $\widetilde{Q}$ in the short run, although for τ small, each row of $\tilde{Q}$ would converge to the stationary distribution.

> *Survival Rate of Bidders* (r): We estimate this parameter using a censored negative binomial model fit to the likelihood of bidder exit using the full sample of bids. A bidder exits if they do not return to bid again. However we treat this outcome as censored if the last bid event was in the final six weeks of our dataset.[40]
>
> *Supply* (π): $\hat{\pi}$ is the observed market share of each of the six resolution types.

Bid normalization. Cameras differ in both observable and unobservable features. The first step in working with the bids is to normalize out the contribution of observables for

[40]We maintain the assumption of exogenous exit on the part of bidders. While it might seem somewhat unrealistic, we found no evidence in reduced–form regressions of a systematic correlation between exit of losing bidders and the level of their bid. However, the assumption could be weakened: see prior versions of this paper, in which we described an extension of our identification results to random outside options.

which bidders have common preferences (Haile et al., 2006). Applying our earlier theory to the valuations in (16), the bidding equation takes the form:

$$b_{i,j,t} = \max\{0, \alpha_{i,c} + \text{res}_j \alpha_{i,r} + Z_{j,t}\gamma + \xi_{j,t} - v(\alpha_{i,c}, \alpha_{i,r})\} \tag{7}$$

i.e. valuation less continuation value. Restricting attention to positive bids, and adding and subtracting $\psi_j \equiv E_{\mathbf{F}_\alpha}[\alpha_{i,c} + \text{res}_j \alpha_{i,r} - v(\alpha_{i,c}, \alpha_{i,r})]$, we get an estimating equation in reduced form:

$$b_{i,j,t} = \psi_j + Z_{j,t}\gamma + e_{i,j,t}$$

where the error term $e_{i,j,t}$ combines an individual-specific deviation in bid from that of the average type and the common unobserved heterogeneity. Since these are both independent of $Z_{j,t}$), we can estimate γ consistently by OLS with product fixed effects. However, since the error here includes bidder resolution types, we use FGLS, allowing for a correlation with camera resolution.

We only run this regression on a subsample of our data, consisting of the two highest bids in every auction. This is motivated by the behavioral intuition of Haile and Tamer (2003). Because eBay's proxy bidding system is formally an ascending auction, the bid of the third, fourth, or n-th highest bidder may not reflect their intended final bid (what they would have bid in a sealed-bid auction i.e. valuation less continuation value). It is common for bidders to "test out" a sequence of ascending bids to see if they can become the standing high bidder, but if in this process the price comes to exceed their intended final bid then it will be censored from the dataset. For this reason we restrict attention to the two highest bids in each auction, who are never censored in this way.

As controls $Z_{j,t}$ we include product line fixed effects, listing attributes including shipping options, seller feedback, and optional listing features (e.g., sellers may pay a fee for their results to be highlighted in search results), as well as a set of dummies for resolution, optical zoom, and digital zoom levels.[41] We report results for the main controls of interest in Table B-3. They are generally sensible, with bids increasing in seller feedback and higher when there is free shipping, consistent with the findings of Einav et al. (2015). Curiously, they are also slightly increasing in shipping costs (where applicable); we believe that this is an quirk of the fact that at the time, shipping costs were not counted towards the final value fee, and

[41]Note that we include resolution dummies to avoid omitted variable bias, however we exclude these coefficients when predicting the normalized bids below.

Table B-3: First-Stage Normalization Regression

	(1) Winning bid
Free shipping	16.65***
	(2.119)
Shipping cost	0.367***
	(0.0496)
Seller feedback (thousands)	0.0447***
	(0.00876)
R^2	0.747
N	16484

Notes: This table presents selected coefficients from the first-stage normalization regression. Unreported here, the regression also includes dummies for all (rounded) resolution, optical zoom, and digital zoom values, as well as product-level fixed effects. It also includes a large array of listing attributes such as featured listing status, whether the seller paid for a scheduled end-time, and dummies for bundled accessories. Note that the coefficients on the resolution dummies are set to zero before predicting the normalized bids for subsequent analysis. Standard errors in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

so sellers has an incentive to shift some of the price to shipping costs.

Define the normalized bids $\widetilde{b}_{i,j,t} = b_{i,j,t} - Z_{j,t}\gamma = \alpha_{i,c} + \text{res}_j \alpha_{i,r} - v(\alpha_{i,c}, \alpha_{i,r}) + \xi_{j,t}$. We will work with our estimate of these bids $b_{i,j,t} - Z_{j,t}\hat{\gamma}$ in what follows. Consider the difference between two normalized bids on the same product, by the same bidder in two different auctions. These should differ only by $\xi_{j,t_1} - \xi_{j,t_2}$, and since these random variables are independent of each other, we can estimate the variance of $\xi_{j,t}$ as $\sigma^2_{\xi,j} = Var(b_{i,j,t} - b_{i,j,t'})/2$ where the RHS variance is estimated by pooling over all $t < t'$ pairs available.[42]

Opposing bids. Here, we estimate $\{\check{G}^1_j\}$ and $\{\check{G}^2_j\}$, the distributions of the highest two competing bids net of unobserved heterogeneity ($\xi_{j,t}$). We observe the raw distributions of the first and second highest normalized bids $\{G^1_j, G^2_j\}$ (i.e. the convolution of $\check{b}_{i,j}$ and $\xi_{j,t}$). Based on the shape of these, we assume a Gamma distribution for $\{\check{G}^1_j, \check{G}^2_j\}$ with separate shape and scale parameters for each product and order statistic. We estimate these

[42] As an additional check, we measured the heterogeneity of bids by the same bidder *on auctions with the same title*, and found it to be substantially smaller, with a standard deviation of 4.68. This suggests that most of the within-bidder, within-resolution variation is driven by unobserved heterogeneity rather than learning, deadlines, or market fluctuations.

parameters by method of moments (i.e. we pick these parameters to match the mean and variance of the distribution of the normalized highest bid, taking our estimates of $\sigma^2_{\xi,j}$ in the earlier step as given).

We summarize the results of all these preliminary estimation steps in Table B-4. We find that bidders are reasonably likely to return and bid again upon a loss ($r = 0.38$). The modal camera is in the middle of the range of resolution types (7MP with 39.71 percent of auctions), while 5MP and 10MP cameras are somewhat underrepresented (12.3 percent and 7 percent, respectively). Finally, unobserved heterogeneity varies substantially with resolution type and is particularly important for 10MP cameras, where we anticipate there is more differentiation of high-end products.

B.6 Estimating Bid Functions and Selection Probabilities

We start with the optimal bid function. Fix a type $\boldsymbol{\alpha}$ and an associated J-vector of valuations $\mathbf{x}$. We solve for their optimal bids and associated continuation value by value iteration: given a (scalar) continuation value $v(\mathbf{x})$, define the following mapping from $\mathbb{R}^+$ into itself:

$$T_{\mathbf{x}}(v) = \frac{\pi \cdot (G^1(\mathbf{x} - v)(\mathbf{x} - \mathbb{E}_{G^1}[B^1 | B^1 \leq \mathbf{x} - v]))}{1 - r\left((1 - G^1(\mathbf{x} - v)) \cdot \pi\right)}$$

where $\cdot$ denotes dot product, and G^1 and E_{G^1} are now J-vectors. The continuation value for type $\mathbf{x}(\boldsymbol{\alpha})$ satisfies $T_{\mathbf{x}}(v) = v$, and from this we obtain the bidding function $\beta(\alpha) = \max\{0, \mathbf{x}(\alpha) - v(\alpha)\}$.

Next, given a bid $\mathbf{b}$ associated with the type $\boldsymbol{\alpha}$, we work out the probability that a particular bidder is observed in a subset $\mathcal{B} \subseteq \mathcal{J}$ of the set of possible auctions. The building block for this is the function $\mathbb{P}(\mathcal{A}, \mathbf{b})$, defined as the probability that a bidder with bid vector $\mathbf{b}$, entering in a randomly sampled period, exits before they are "observed" in any auction outside of the set $\mathcal{A} \subseteq \mathcal{J}$. Define $\mathbb{P}(\mathcal{A}, \mathbf{b}, j)$ in the same way, but additionally conditioning on the bidder entering in state j, so that $\mathbb{P}(\mathcal{A}, \mathbf{b}) = \sum_j \pi_j \mathbb{P}(\mathcal{A}, \mathbf{b}, j)$. Recalling that we treat bidders as "observed" only when they make bids that qualify for our estimation sample (i.e. one of the top two bids in the auction), we can write the latter probability recursively as:

$$\begin{aligned} \mathbb{P}(\mathcal{A}, \mathbf{b}, j) = {} & 1(j \in A)\left(G^1_j(b_j) + (1 - G^1_j(b_j))(1 - r + r\mathbb{P}(\mathcal{A}, \mathbf{b}))\right) \\ & + 1(j \notin A)\left((1 - G^2_j(b))((1 - r) + r\mathbb{P}(\mathcal{A}, \mathbf{b}))\right). \end{aligned} \quad (8)$$

Table B-4: First-Stage Parameters

r				0.3806		
				(0.0055)		
π		5mp	6mp	7mp	8mp	10mp
		0.1232	0.3326	0.3971	0.0766	0.0704
		(0.0034)	(0.0049)	(0.0051)	(0.0028)	(0.0027)
σ_ξ		5mp	6mp	7mp	8mp	10mp
		19.0497	12.5239	12.5960	8.6771	20.4836
		(0.1715)	(0.0704)	(0.0552)	(0.1622)	(0.1715)
$G^{(1)}$		5mp	6mp	7mp	8mp	10mp
	k	23.1098	36.8410	27.6030	34.9949	79.5150
		(1.4572)	(1.3320)	(0.7670)	(1.7766)	(6.8781)
	θ	6.6671	5.6896	8.4362	7.4766	4.2940
		(0.4277)	(0.2088)	(0.2438)	(0.3845)	(0.3841)
$G^{(2)}$		5mp	6mp	7mp	8mp	10mp
	k	21.6889	37.9918	27.6053	31.2232	107.5733
		(1.4823)	(1.5179)	(0.8154)	(1.5900)	(11.6799)
	θ	6.6180	5.2130	8.0352	7.8953	3.0116
		(0.4539)	(0.2103)	(0.2468)	(0.4072)	(0.3348)

Notes: This table presents estimates of the first-stage parameters of the model. See text for estimation details. Standard errors are presented in parentheses.

On the first line of the RHS we consider the case where $j \in \mathcal{A}$: that bidder will be observed only in $\mathcal{A}$ if they exit immediately (either by winning or exogenously exiting), or survive and then transition to a new random state, in which case the chance is $\mathbb{P}(\mathcal{A}, \mathbf{b})$. The second line is for the case where $j \notin \mathcal{A}$, and then the bidder must *not* be observed in j, so they must make a bid below the second highest. Given that this is the case, again they must either exit immediately or survive and we get $\mathbb{P}(\mathcal{A}, \mathbf{b})$ again.[43]

Now G_j^1 and G_j^2 are known, and $\mathbb{P}(\mathcal{A}, \mathbf{b})$ is a weighted sum of the $\mathbb{P}(\mathcal{A}, \mathbf{b}, j)$, so for any $(\mathcal{A}, b)$ we can get $\mathbb{P}(\mathcal{A}, \mathbf{b})$ as the solution to a linear system with unknowns $\{\mathbb{P}(\mathcal{A}, \mathbf{b}, j)\}_{j\in\mathcal{J}}$. Then we can use these objects to compute the probability that a bidder is observed in *exactly* the set $\mathcal{B}$. For instance, if we are interested in the probability that a bidder of type $\boldsymbol{\alpha}$ is observed in a subset $\mathcal{B} = \{2, 4\}$ we can compute $\mathbb{P}(\{2, 4\}|\beta(\boldsymbol{\alpha})\} = \mathbb{P}(\{2, 4\}, \beta(\boldsymbol{\alpha})) - \mathbb{P}(\{2\}, \beta(\alpha)) - \mathbb{P}(\{4\}, \beta(\alpha)) + \mathbb{P}(\{\phi\}|\beta(\alpha))$.

B.7 Optimization

We now proceed to the problem of optimizing the likelihood. Usually one would proceed by applying some optimization routine in an outer loop to the likelihood (17), evaluating it by Monte Carlo integration, holding the randomness in the draws of α fixed across evaluations k (Pakes and Pollard, 1989). But this is computationally costly, as for many different α draws we will have to evaluate the bidding and selection functions.

Notice that the parameter vector θ enters the likelihood (17) only through the distribution of random coefficients $F(\alpha|\theta)$. This allows us to employ an alternative approach based on re-weighting (Ackerberg, 2009). Fixing α, define

$$h_i(\alpha) \equiv \frac{\mathbb{P}\{\mathcal{B}_i|b(\alpha)\}}{\mathbb{P}\{|\mathbb{P}\{\mathcal{B}_i| = 2|b(\alpha)\}} \left(\prod_{j\in\mathcal{B}_i} \phi\left(\frac{b_i - b_j(\alpha)}{\sigma_{\xi,j}}\right)\right) |\beta'(\alpha)|, \tag{9}$$

and note that this object is computable for any α from the results of the first-stage estimation. Therefore, taking as our objective function the log-likelihood of the dataset, we can write

$$L(\theta, \{b_i\}) = \sum_i \log \int h_i(\alpha) dF(\alpha|\theta). \tag{10}$$

One natural and computationally efficient way to proceed would then be to sample α uniformly from the type space S times (for S the number of draws in the Monte Carlo integration), and then optimize the simulated criterion function by re-weighting their likelihood

[43] This calculation assumes that the true state space coincides with the one that bidders use in forming beliefs. This is not necessary (we could condition on history), but is practical given our data constraints.

contribution according to $f(\alpha|\theta)$. Instead, we choose to use importance sampling, i.e. sampling from and re-weighting relative to a user-chosen distribution, denoted $G(\alpha)$. We do this for two reasons: first, for efficiency—we would like to sample more points from the center of the true distribution if possible, since this is where the gradient of the likelihood is sharpest. Second, we would like the sampled distribution to have unbounded support, mirroring our parametric assumption on $F(\alpha|\theta)$.

Let the set of points be drawn from $G(\alpha)$; then we have:

$$\hat{\theta} = \arg\max_{\theta} \; L(\theta, \{b_i\}) = \sum_i \log \sum_{s=1}^{S} h_i(\alpha) \frac{f(\alpha_s|\theta)}{g(\alpha_s)}. \tag{11}$$

We implemented the estimator in two steps: first, we choose G_1 to be a normal with very large variance, and obtained a first estimate $\tilde{\theta}$. In a second step, we drew from a distribution $G_2(\alpha) = F(\alpha|\tilde{\theta})$ and re-optimized. This two-step procedure should improve the efficiency of our estimator, as the second sample should be more centered and thus provide a better approximation to the integral.

B.8 Consumer Surplus

Even without estimating the full demand system, at this point we can already say something useful about consumer surplus. In the static model consumers bid their valuations whereas in our dynamic setting we have shown that they shade their bids substantially. This shading will bias static estimates of consumer surplus.[44]

In the static framework where $b_j = x_j$, the difference between the highest bid and the second-highest bid is a direct measure of surplus per auction. So for a psuedotype $\mathbf{b}$ entering a random auction according to π,

$$\mathbb{E}[CS_{static}(\mathbf{b})] = \sum_j \pi_j \{G_j^1(b_j)[b_j - \mathbb{E}[B^{(1)}|b^{(1)} < b_j]]\}$$

To compute this object we take total surplus to be the total sum of $\{B^{(1)} - B^{(2)}\}_i$, i.e. the valuation of the first-highest bidder minus the price paid, across all auctions, and then divide

[44] Coey et al. (2015) also make a point of computing consumer surplus in a dynamic framework, although theirs is based on valuations with "deadlines."

by the total number of bidders. From Table B-1 the average difference between the first– and second–highest bid in our dataset is \$8.90, which yields an expected consumer surplus of (\$8.90 × (9,213 auctions ÷ 38,919 bidders) =) \$2.11 per bidder in our dataset.

In the dynamic model, however, bidders shade their bids: $b_j = x_j - r\sum_k \tilde{Q}_{j,k} v_k(\mathbf{b})$. Note moreover that, if we take bidders' beliefs to be correct, $v(\mathbf{b}) = \mathbb{E}[CS_{dynamic}(\mathbf{b})]$.[45] In this setting,

$$\begin{aligned}
\mathbb{E}[CS_{dynamic}(\mathbf{b})] &= \sum_j \pi_j \{G_j^1(b_j)[x_j - \mathbb{E}[B^{(1)}|b^{(1)} < b_j]] + (1 - G_j^1(b_j))r\sum_k \tilde{Q}_{j,k} v_k(\mathbf{b})\} \\
&= \underbrace{\sum_j \pi_j \{G_j^1(b_j)[b_j - \mathbb{E}[B^{(1)}|b^{(1)} < b_j]]\}}_{=\mathbb{E}[CS_{static}(\mathbf{b})]} + \underbrace{\sum_j \pi_j r \sum_k \tilde{Q}_{j,k} v_k(\mathbf{b})}_{=r\mathbb{E}[CS_{dynamic}(\mathbf{b})]} \\
&= \frac{1}{1-r}\mathbb{E}[CS_{static}(\mathbf{b})].
\end{aligned}$$

Note that in the second equality, we take advantage of the fact that π is the ergodic distribution of Q, and therefore $\tilde{Q}\pi = \pi$ for any τ.

This computation yields a substantially larger estimate of consumer surplus per bidder: ($(0.3806)^{-1}$ × \$2.73 =) \$5.54 (\$23.38 per auction). This large correction reflects the fact that the bidders who are most likely to win also shade the most, and therefore it is for these that the static model most underestimates valuations.

[45] This relies on an assumption that τ is sufficiently small that each time a bidder participates, sufficient time has passed that the market is once again in steady-state i.e. $v(\mathbf{b})$ can be calculated without sophisticated conditioning on histories. We could be more mathematically precise—as we were in the earlier selection correction argument—but as we argued when discussing Table B-1, τ appears to be small.